Physics for Physiotherapy Technology

203-9P1-DW Fall 2022

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Don’t Panic!

Yes, this is a physics course. But if you understand ***why*** you are taking it, then you’ll see that it’s not such a bad thing. Even better, you’ll know what you need to get out of this course to prepare for the later courses in your program.

First: I do not need you to become a physicist! Physics comes by its bad reputation honestly. Describing the universe precisely can be done, but most times requires extremely complicated mathematics and abstract thinking. That is true in cases like exploring what happens when you fall into a black hole, or asking about the quantum mechanics of the atom. But we are not going to consider such esoteric things! This course is about getting you ready to become a Physiotherapy Technologist. Our focus is the human body.

Second: What you learn in this course is structured to prepare you for the courses you will be taking later in your program. You need to know about forces (pushing), torques (twisting), and materials (bending) to prepare for reasoning about *Biomechanics*. You need to know about energy (heat), waves (sound), and electricity to prepare for the use of *Electrotherapy*. The focus is not on the mathematical descriptions. Yes, to make things concrete there will be math, but our focus will be on how to think about these core ideas *conceptually*. I do not need you to become a physicist, but I do need you to learn how to think *physically*.

Third: This text is meant for you to read! Reading this *before* class will prepare you for what we will be *doing* in class. It is not expected that you will gain a complete understanding just by reading. What is expected after your first reading of any topic is that you will have questions. Read this, and then bring those questions to class. There we will work together to understand and answer them.

– Chris Roderick
2019 August 21
Updated 2022 August 18

Contents

Chapter 0

Preparation

Yes, this is chapter “zero”. This is not a course about math, but we will use math. So, before we get to chapter 1 (where we’ll start to think about Forces), we need to review a few topics. These will be things that you have seen (or, at least, were supposed to have seen) in your high school math and science courses:

- Units
- Trigonometry
- Vectors

The “units” I’m talking about here are the standard S.I. units of metres and kilograms. We need to remember what units are appropriate for which quantities, and we need to know how to *convert* units. “Trigonometry” means the mathematics of triangles and angles, and the functions sine, cosine and tangent. That math will be needed for when we work with vectors, which describe forces.

One other topic you may or may not have seen is

- Logarithms

The logarithm function will be used when we explore sound and how we measure “loudness”, in chapter 5.

There will be many practice problems here. It is not expected for you to finish them all before we move on to the other topics in the course, so do make a point of coming back here to work on them incrementally, or to re-do them as you find necessary.

0.1 Units of Measurement

Back in the 1970’s Canada began the move away from the Imperial System of Measurement and towards using Le Système International d’Unités (sometimes referred to as the Metric System, but commonly known as the SI). By the mid-1980’s the public school systems and most governmental organizations had almost completely made the shift. But Canadian society, as a whole, can not be exclusively Metric. Because our largest trading partner, the United States of America, uses the Imperial system of measurement our industrial and public sectors can not completely move away from the use of inches and feet, and ounces and pounds. While the fundamental system of measurements used in this text will be the SI, we can not afford to be ignorant of the Imperial system.

0.1.1 What does it mean to measure?

To measure something is to compare it to a standard. Lengths are determined by placing the thing next to a calibrated ruler. Mass is determined by balancing the thing with an equal amount of calibrated masses. A time is measured by noting the coincidence of an event with a “tick” on a clock.

The result of measuring something is usually to state what multiple of that standard the thing is. “How thick is this book?” Three and a half centimetres. “How heavy is that bicycle?” Eighteen kilograms. “When does this class begin?” In six minutes.

CRITICAL : Include the Units

> A quantity without units has no meaning.

If a measurement of a quantity is made using two different units, the measurements will have different numerical values. For example, if I measure the mass of my cellphone using kilograms and using pounds, I get the values 0.170 kg and 0.375 lb. Even though the numerical values are different *these measurements are the same*. A pound of mass is less than half a kilogram. Using a smaller unit gives a larger number. This is so important that I will say it again:

Using a smaller unit gives a larger number.

Remember this whenever you do any units conversion. It can help you check your result.

0.1.2 Length

If I ask you to measure something, you would probably reach for your ruler. Length is not the only physical quantity we can measure, but I think that is the first one that most people associate with the idea of “measuring something”.

If I ask you to measure the length of your cellphone with a ruler, you might answer me with something like “15.8 cm”. If I ask someone else to measure the length of your cellphone using their metre stick, they might answer me with something like “0.16 m”. If we later refer to the manufacturer’s website we would find that the length of the cellphone is specified as 158.2 mm. We each may have each found slightly different numerical values, but the important feature is that these values do not disagree, that they are consistent with each other.

Each of our measurements of length, made with Metric units, are expressed as a multiple of a ***metre***. The metre is the *base unit* of length in the Metric system. Lengths, areas, and volumes are expressed as multiples and products of the metre. (Notice the spelling: a met***re*** is the base unit of length in the Metric system; a met***er*** is the noun used to refer to a device for measuring something, like an ammeter for measuring electrical current.)

Conversion between SI measurements of length involve powers of ten. Specifically, the primary multiples of length in the SI are related by multiples of one thousand ($1000 = 10 \times 10 \times 10 = 10^{+3}$). One kilometre is one thousand metres ($1\,\text{km} = 1 \times 10^{+3}\,\text{m} = 1000.\,\text{m}$, exactly). Kilometres are large enough to be useful for measuring distances across the Earth, for example, between cities. One millimetre is a thousandth of a metre ($1\,\text{mm} = 1 \times 10^{-3}\,\text{m} = 0.001\,\text{m}$, exactly).

Two other multiples you may have heard of are the micrometre ($1\,\mu\text{m} = 1 \times 10^{-6}\,\text{m}$) and the nanometre ($1\,\text{nm} = 1 \times 10^{-9}\,\text{m}$). The micrometre is used in engineering to specify the precise sizes of machine parts. The nanometre is scale at which the circuitry in computer chips is formed. In the biological context, a human red blood cell is about eight micrometres in diameter, while the size of the hemoglobin molecules that cell carries are each about seven nanometres in diameter.

The most-used exception to that pattern of thousands is the ***centimetre***. A metre is too large a unit for objects that a person might manipulate with their hands. Most cellphones are much smaller than a metre in size. Sandwiches, too. So a more reasonable, more human-size unit is the centimetre, which is a hundredth of a metre ($1\,\text{cm} = 1\times10^{-2}\,\text{m} = 0.01\,\text{m}$, exactly). By this definition there are ten millimetres in a centimetre ($10.\,\text{mm} = 1.0\,\text{cm}$).

name:	nano	micro	milli		kilo	mega	giga
symbol:	n	µ	m		k	M	G
power:	10^{-9}	10^{-6}	10^{-3}	1	10^{+3}	10^{+6}	10^{+9}

These prefixes apply to all SI units, not just length. The table here shows the range that will be of use to use in this course. (There are other larger and smaller prefixes, but they are of no use in this course.) Being able to express these prefixes as the correct power of ten, and being able to convert from one prefix to another, is an important skill to develop.

Example 0.1 : That's a lot of change . . .

The Canadian one-dollar coin (the "Loonie") is a disk of diameter 26.5 mm and thickness 1.95 mm. A "roll" of Loonies is $50 stacked in a cylinder, measuring $50\times1.95\,\text{mm} = 9.75\,\text{cm}$ from end-to-end. If you could stack Loonies into a cylinder one kilometre long, how much money would that be?

To find the amount of money, we need to find the number of Loonies. The number of Loonies times the thickness of a single Loonie must equal a kilometre ($10^{+3}\,\text{m}$). Since each coin is on the order of a millimetre ($10^{-3}\,\text{m}$) we should not be surprised if we have something like a million dollars!

The relation "the number of Loonies times the thickness of a single Loonie equals a kilometre" has the equation

$$n\times1.95\,\text{mm} = 1.00\,\text{km} \tag{1}$$

$$n = \frac{1.00\,\text{km}}{1.95\,\text{mm}} = \frac{1.00\times10^{+3}\,\text{m}}{1.95\times10^{-3}\,\text{m}} = 5.13\times10^{+5} \tag{2}$$

This is a little more than half a million coins, with an approximate value of five hundred and thirteen thousand dollars.

Important: To evaluate the numerical value of the ratio of lengths we had to express both in terms of the common unit of metres. If we had left the lengths in their original units (km & mm) our "answer" would have been 0.513. That would have said that a kilometre-long stack of Loonies was worth half a dollar, which is clearly *nonsense*. Remember to always be explicit about your units in your calculations.

Imperial Lengths

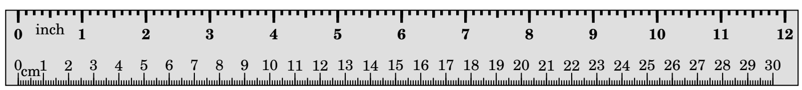
There are many, many different units of length in the Imperial system. For the purposes of this course there are only two that we will be using: the inch and the foot. These are important as they are the two Imperial units that might used in the measurement of patients' bodies, and of human-scale objects like furniture, buildings and vehicles.

Since 1930 the inch has been defined to be

$$1\,\mathrm{in} = 25.4\,\mathrm{mm} \tag{3}$$

The value 25.4 mm is an exact value, not a number with only three significant figures. This redefinition was done to insure that the inch standard of measure could be reproduced accurately. The subsequent accuracy follows from the precision with which metric distances can be determined and reproduced scientifically. The feature of this definition that is important to us is that the inch is larger than a centimetre. Consequently measurement of length in centimetres will have a larger value than when measured in inches.

A foot of distance is defined to be 12 in exactly. Due to the definition of the inch, a foot is 30.48 cm exactly. If you ever wondered why your school ruler was a 30 cm ruler, wonder no more: Companies used to make foot-rulers and it was cheaper to keep making rulers the same length, using the same amount of material, and just change the markings on them!



This ruler is not to scale. Sorry.

Lengths measured in Imperial units that are greater than twelve inches are usually written as the sum of a length in feet and a length in inches. For example, a length of 55 inches is equivalent to

$$55\,\mathrm{in} = 4 \times 12\,\mathrm{in} + 7\,\mathrm{in} = 4\,\mathrm{ft} + 7\,\mathrm{in} \tag{4}$$

which is written 4ft 7in, or more conventionally as $4'\,7''$.

Example 0.2 : What's that measured in . . . ?

A patient tells you they are 5ft 10in tall. What is that in metres? You ask them to try lifting a hand to height of 2.00 m above the floor. What is that measured in feet and inches?

We know that 30 cm is approximately 1 foot. The patient's height is almost 6 feet, which should be about 180 cm. Going back the other way, 200 cm should be a little less than 7 feet.

To convert feet and inches to metric begin by expressing the imperial measurement entirely as a multiple of inches:

$$5\,\mathrm{ft} + 10\,\mathrm{in} = 5 \times 12\,\mathrm{in} + 10\,\mathrm{in} = 70\,\mathrm{in} \tag{5}$$

Then convert the number of inches to centimetres, then to metres:

$$70\,\mathrm{in} = 70 \times 2.54\,\mathrm{cm} = 178\,\mathrm{cm} = 1.78\,\mathrm{m} \tag{6}$$

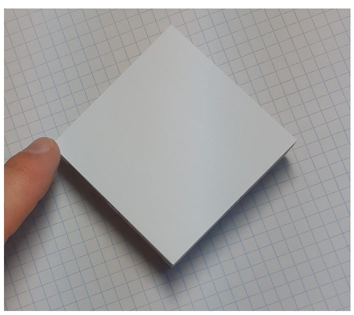
To convert the metric measurement to feet and inches, reverse the steps taken above:

$$2.00\,\mathrm{m} = 200\,\mathrm{cm} = 200 \times (\tfrac{1}{2.54}\,\mathrm{in}) = 78.7\,\mathrm{in} = 6 \times 12\,\mathrm{in} + 6.7\,\mathrm{in} = 6\,\mathrm{ft} + 7\,\mathrm{in} \qquad (7)$$

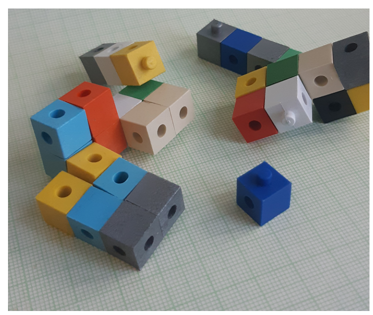
This is only 9 inches above their head. Unless they have a limited range of motion in their shoulder, they should be able to reach this height.

There are two other Imperial units of length that might have heard of: the yard and the mile. A yard of distance was defined to be three feet, which is slightly less than a metre ($3\,\mathrm{ft} = 3 \times 12 \times 2.54\,\mathrm{cm} = 91.44\,\mathrm{cm}$). A mile of distance was defined to be 5280 ft. Due to the current definition of the inch in terms of the millimetre (1 in = 25.4 mm) a yard of distance equals 0.9144 m and a mile of distance is now defined to be 1609.344 m. For quick reference you can think that a yard is about one metre. Since a mile is a little more than a kilometre and a half (1.6 km), distances in miles will be smaller numbers than measured in kilometres. These two units are really only of interest if you are reading or watching news or sports from the US.

0.1.3 Areas & Volumes



Like a square metre, only smaller.



Hello, old friend.

In the photograph above on the left is a paper square that measures 7.4 cm by 7.4 cm. If it were a square measuring 1 m by 1 m then we would say that its ***area*** was one square metre, written as $1\,\mathrm{m}^2$. (Notice carefully the exponent over the symbol for the unit of length.) So what fraction of a square metre is piece of paper in the photograph?

For a flat surface, you know the rule: the area is the width multiplied by its height. The area of the square in the photograph, measured in square metres, is

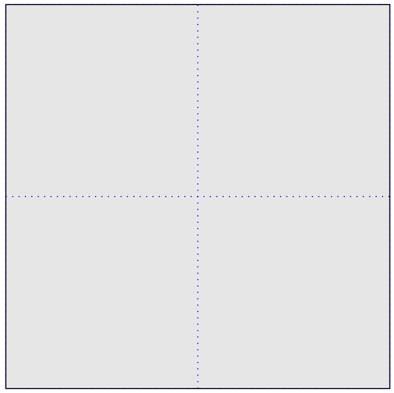
$$(7.4\,\mathrm{cm}) \times (7.4\,\mathrm{cm}) = (0.074\,\mathrm{m}) \times (0.074\,\mathrm{m}) \qquad (8)$$

$$= (0.074 \times 1\,\mathrm{m}) \times (0.074 \times 1\,\mathrm{m}) \qquad (9)$$

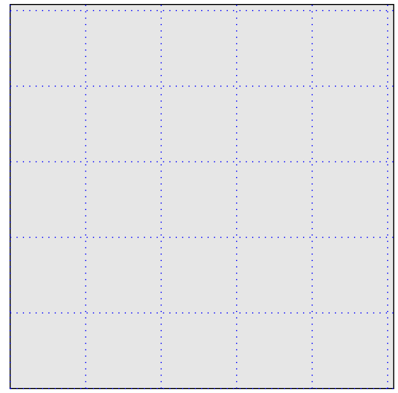
$$= (0.074 \times 0.074) \times (1\,\mathrm{m} \times 1\,\mathrm{m}) \qquad (10)$$

$$= 0.005476 \times 1\,\mathrm{m}^2 \qquad (11)$$

Take your time, and follow the steps in the calculation above. Since our goal is to find the area in square metres begin by converting all lengths to metres. Then separate the numerical factors from the units. The last step is to recognize that a square metre ***is*** the quantity "$1\,\mathrm{m} \times 1\,\mathrm{m}$".



A 2in-by-2in square, divided into square inches.



A 2in-by-2in square, divided into square centimetres.

The picture on the left makes it clear that $2\text{ in} \times 2\text{ in} = 4\text{ in}^2$. The way that you should think about areas is like this:

$$4\,\text{in}^2 = 4 \times \{(1\,\text{in}) \times (1\,\text{in})\} \tag{12}$$

Our measurement of the area is four increments, where each increment is a square that measures one inch by one inch. What is this $4\,\text{in}^2$ area when measured in square centimetres? The picture on the right, where the area is divided up into square centimetres, makes it clear that there are more cm^2 than in^2 since they are smaller. (Remember: smaller unit, larger number.)

When we need to convert units of area, we must convert *each* of the units of *length* that combine to make the unit of area. To follow this, I will color-code the pieces. With $1\text{ in} = 2.54\text{cm}$, we get

$$
\begin{aligned}
4\,\text{in}^2 &= 4 \times \{(1\,\text{in}) \times (1\,\text{in})\} && (13)\\
&= 4 \times \{(2.54\,\text{cm}) \times (2.54\,\text{cm})\} && (14)\\
&= \{4 \times 2.54 \times 2.54\} \times \{(1\,\text{cm}) \times (1\,\text{cm})\} && (15)\\
&= 25.8 \times \{1\,\text{cm}^2\} && (16)\\
&= 25.8\,\text{cm}^2 && (17)
\end{aligned}
$$

Contrast this process of converting areas with the conversion of the length $4\text{ in} = 4 \times (2.54 \times 1\,\text{cm}) = 10.16\,\text{cm}$.

The same process must be followed when we convert between square metres and square centimetres we must convert *each* of the units of *length*. For example:

$$
\begin{aligned}
8\,807\,\text{cm}^2 &= 8\,807 \times (1\,\text{cm}^2) && (18)\\
&= 8\,807 \times (1\,\text{cm} \times 1\,\text{cm}) && (19)\\
&= 8\,807 \times (0.01\,\text{m} \times 0.01\,\text{m}) && (20)\\
&= 8\,807 \times (0.01 \times 0.01) \times (1\,\text{m} \times 1\,\text{m}) && (21)\\
&= 0.8807 \times 1\,\text{m}^2 = 0.8807\,\text{m}^2 && (22)
\end{aligned}
$$

If we were converting a length in centimetres into a length in metres, then we would only move the decimal over two places. But, for an area we need to convert both the length and width, and we move the decimal over two and then two again (a conversion for each of the length and the width).

The Litre

The metric unit of volume is the *litre*. It is defined to be the volume of a cube whose sides are 10 cm each. (Get out your ruler and visualize a cube with this volume!) This volume is a small fraction of a cubic metre: $(10\,\text{cm})^3 = (0.10\,\text{m})^3 = 0.001\,\text{m}^3$.

The symbol for the litre is upper-case L. (Sometimes you might see the symbol "ℓ" being used; but we will use L.) Because of this definition a cubic meter has a volume of 1000 L, and a cubic centimetre has a volume of on millilitre:

$$1\,\text{cm}^3 = 1\,\text{mL} \tag{23}$$

As with conversions of areas, when we convert units of volume we must convert each of unit of *length* that combines to make the unit of volume.

Example 0.3 : Converting volumes

How many cubic centimetres are in a 2.7 L volume?

The relation between the number n of cubic centimetres and the volume given is

$$n \times 1\,\text{cm}^3 = 2.7\,\text{L} \tag{24}$$

Isolating the number n we can then calculate by expressing all quantities in terms of base SI units:

$$n = \frac{2.7\,\text{L}}{1\,\text{cm}^3} = \frac{2.7 \times (10^{-1}\,\text{m})^3}{1 \times (10^{-2}\,\text{m})^3} = \frac{2.7 \times 10^{-3}\,\text{m}^3}{1 \times 10^{-6}\,\text{m}^3} = 2.7 \times 10^{+3} \tag{25}$$

This shows that 2.7 L is equal to 2 700 cubic centimetres.

Another, more direct, way to do this conversion is to express the litre directly in terms of cubic centimetres:

$$2.7\,\text{L} = 2.7 \times (10\,\text{cm})^3 = 2.7 \times (10^{+3}\,\text{cm}^3) = 2\,700\,\text{cm}^3 \tag{26}$$

A cubic centimetre ($1\,\text{cm}^3$) is often given the symbol "1 cc". The preceding example shows that one cubic centimetre is equal to a volume of one-thousandth of a litre, the millilitre: 1 cc = 1 mL.

■**PICTURE:** photo: 2L pop bottle inside a cubic metre

While the gramme was too small a unit of mass (leading to the choice of the kilogram for the base unit), a cubic metre is too large a unit of volume for day-to-day human-scaled applications. A two-litre bottle of carbonated beverage is a size you are probably familiar with. Calling this volume "2 L" is more practical than calling it "$0.002\,\text{m}^3$".

0.1.4 Mass

Mass is the physical property of an object that relates to the amount of material that composes it. Mass is not size, as a cube of aluminum and a cube of iron of identical sizes will have different masses (the iron having almost three times as much mass).

The gramme (in modern usage written gram) was defined to be the mass of a cubic centimetre of water that is just above its freezing point. If you just look at a cubic centimetre you can appreciate that it is not really large, and that a gram would be a really small unit of mass. For practical reasons the ***kilogram*** became the base unit of mass in the Metric system. This also corresponds to the mass of a 1 L volume of water. (For reference, note that a 2 L container of cola has a mass of 2 kg, approximately. This is a useful number to keep in mind when trying to grasp the size of a mass you might have calculated.)

Imperial Masses

One of the imperial units of mass is the pound, defined to be 0.453 592 37 kg exactly. This means that 1 kg is approximately 2.2 pounds. The origins of the pound as a unit of mass are traced back to the ancient Roman empire, where the unit was then called the "libra". For this reason the unit symbol for the pound is "lb".

Although the kilogram is the base unit of mass, it is most likely that any patient you treat will know their mass in pounds but not in kilograms. This is a vestige of the transition from imperial units to metric happening in their lifetime. Consequently knowing how to convert between pounds and kilograms may be necessary for the purposes of collecting and interpreting patient data. Since the pound is a smaller unit than the kilogram a mass measured in pounds will have a larger numerical value than the same mass measured in kilograms. (For reference note that 100. lb = 45.4 kg, and 220. lb = 100. kg, approximately.)

"Weight"

The weight of an object is defined to be the amount of *force* required to hold it up against gravity. In your day-to-day life weight and mass are proportional to each other, and that proportionality is a constant: $w = mg$. For this reason in day-to-day, non-scientific contexts a person's mass is usually referred to as their "weight", and "weighing" someone is to measure their mass. (This is technically incorrect, but everyone understands it by the context.) We will study the difference between mass and weight in more detail in sub-section 1.2.4.

0.1.5 Time

Changes take time to happen. The base unit of measurement of time is the ***second***. The second is Metric. But the minute, hour and day are not. These inter-relationships you should know and have memorized:

$$1\,\text{minute} = 60\,\text{s} \tag{27}$$

$$1\,\text{hour} = 60\,\text{minute} = 3\,600\,\text{s} \tag{28}$$

$$1\,\text{day} = 24\,\text{hour} \tag{29}$$

Converting between hours, minutes and seconds will be done quite frequently in this course – be certain to practise those conversions. The larger units of time (the week, month and year) will not be used very often. There are three common abbreviations used when measuring intervals of time:

$$1\,\text{min} = 1\,\text{minute} \tag{30}$$
$$1\,\text{h} = 1\,\text{hour} \tag{31}$$
$$1\,\text{d} = 1\,\text{day} \tag{32}$$

It will be important to watch for the context when reading or using these symbols so that they do not get confused for the “min” function of mathematics, and the Metric prefixes for “hecta” and “deca”, respectively.

The second is the base unit of time in the Metric system. But, in the context of *Physiotherapy*, times are often measured in minutes and hours. To perform calculations while handling units correctly it is almost always required that we express time intervals in seconds. We will practise conversions of this type in class. The importance of the correctness of the units used to measure time is manifest in the next subject: *rates*.

0.1.6 Rates

The relation that defines a *rate* is

$$\text{change} = \text{rate} \times \text{time} \tag{33}$$

If a quantity changes, and we have a measure of the time it took for the change to happen, then we can talk about the rate of change. If the changing quantity is position, then the rate is what we call speed, measured in metres per second (or kilometres per hour). If the changing quantity is speed, then the rate is acceleration, measured in metres per second squared (m/s^2). If the changing quantity is an amount of liquid or gas, then we can talk about the rate of mass (kilograms per second) or rate of volume (litres per second). If the changing quantity is energy, then the rate is power, measured in joules per second (which defines the unit *watt*). If the changing quantity is electric charge, then the rate is electric *current* (one of the topics in chapter 6) measured in coulombs per second (which defines the unit *ampere*).

In the chapter on Waves (chapter 5) we will study the physics of oscillation and sound. Oscillation, or repeating motion can be measured two ways: the time between repetitions, or the number of repetitions in an amount of time. The latter choice (repetitions per second) is called the *frequency*. Frequency is measured in units of *repetitions per second*, but a “repetition” is not really a unit in the same way that a metre is unit. But when we write out the terms being used to calculate a frequency it is good practice to include “rep” next to the quantity of repetitions so that we can explicitly check that we are correctly forming a ratio that will result in a frequency.

The unit used to measure frequency is ***hertz***, defined by

$$\frac{1\,\text{rep}}{1\,\text{s}} = 1\,\text{hertz} = 1\,\text{Hz} \tag{34}$$

For example, if an object moves repeatedly back-and-forth thirty-seven times in 1.80 s, then the frequency of the object’s oscillation is

$$\frac{37.0\,\text{rep}}{1.80\,\text{s}} = \frac{37.0}{1.80} \cdot \frac{\text{rep}}{\text{s}} = 20.6\,\text{Hz} \tag{35}$$

We will study phenomena like this in chapter 5.

As with conversion of areas, when we convert rates we must be careful to convert all factors to the base SI units. For rates, which are defined by ratios of quantities, we must convert both the numerator and the denominator. Using the correct unit of time is critical. As a concrete example of this, consider *speed*.

Speed

Velocity is a vector that quantifies the motion of an object. The direction of that vector is the direction the object is moving. The magnitude of that vector is the rate at which the object is moving in that direction. That magnitude is called the *speed*. Velocity is a vector, and its magnitude – the speed – is a number. In the Metric system the units we will be using to measure speed are metres per second (m/s), and sometimes kilometres per hour (km/h).

As an example of the difference between the changes and the rate of change: I shuffle to my kitchen in the morning to get my coffee, travelling 5 metres in 9 seconds. My speed is

$$\frac{5\,\text{m}}{9\,\text{s}} = \frac{400 \times 5\,\text{m}}{400 \times 9\,\text{s}} = \frac{2\,000\,\text{m}}{3\,600\,\text{s}} = \frac{2\,\text{km}}{1\,\text{h}} = 2\,\text{km/h} \tag{36}$$

The point of this example is to notice that, while my speed is 2 km/h, I did not travel 2 km and I did not travel for an hour. I was traveling at a speed that, after an hour, would take me 2 km, but after the 9 s of my trip had only taken me 5 m. Remember this example to clarify the difference between the *rate* at which change is happening and the *amount* of change that has happened.

The conversion between a speed measured in km/h and the same speed measured in m/s is not obvious. Our rule of "bigger unit, smaller number" doesn't help us immediately because the kilometre is bigger than the metre, but the hour is also bigger than the second. So it is not immediately clear if 1 km/h is bigger than or smaller than 1 m/s. The conversion can be determined as follows:

$$1\,\tfrac{\text{km}}{\text{h}} = 1 \times \tfrac{1000\,\text{m}}{3600\,\text{s}} \approx 0.2778\,\tfrac{\text{m}}{\text{s}} \tag{37}$$

And the other way around

$$1\,\tfrac{\text{m}}{\text{s}} = 1 \times \frac{(\frac{1}{1000}\,\text{km})}{(\frac{1}{3600}\,\text{h})} = \tfrac{36}{10}\,\tfrac{\text{km}}{\text{h}} \tag{38}$$

(The answer here is written as a fraction to show that the result "3.6" is an exact number.) From this we can see that km/h is a smaller unit of speed than the m/s. (For reference note that 100. m/s = 360. km/h, and 100. km/h = 27.8 m/s, approximately.)

There are other, similar measures of speed in the Imperial system of measure (feet per second, and miles per hour). You will look at those in some of the exercises.

Rates that aren't Time

The idea of "rate", introduced above, was built on the idea of measuring "how fast"; the idea of the variation with time. Mathematically, we had

$$\text{change} = \text{rate} \times \text{time} \tag{39}$$

But this idea of rate can be generalized. When paying for food, gasoline, or electricity we are used to thinking about price per mass, per volume, or per joule. So, in general, a rate is an *inter-relation* between two quantities:

$$\text{change in A} = \text{rate} \times \text{change in B} \tag{40}$$

(where “A” and “B” are the inter-relating quantities). When we *graph* an inter-relationship we have this definition:

$$\Delta y = \text{slope} \times \Delta x \tag{41}$$

This you have seen in your high school maths. In the context of this course (and in your program) our focus is now on how the *units* are related.

Food items are usually sold on a price per mass basis. For example, on the day that I’m writing this, red onions are being sold for \$5.49/kg at my local grocery store. By law in Canada nutritional information is required on packaged foods. Food, obviously, provides our bodies with chemical energy. On nutrition labelling this is written as the number of calories per mass, per volume, or per “serving”. For example, whole-grain “Goldfish” crackers are 90 calories per 37 crackers (listed as 20 g), which is an energy per mass.

Gasoline is an everyday item that we are used to thinking of in proportion to its volume. On the day that I’m writing this “regular” grade is being sold for \$1.05/L at my local gas station. In a manner similar to nutritional information for food, there is a relation between volume of gasoline and energy released by combustion: 42 MJ/L.

Given two blocks of equal volume, one made of wood and one made of iron, the block of iron will have a greater mass. The proportionality between volume and mass is called *density*, and is symbolized by the Greek letter rho ρ:

$$\text{mass} = \text{density} \times \text{volume} \tag{42}$$

$$m = \rho V \tag{43}$$

Density is a characteristic property of an object, one that usually lets us identify what material it is made of. The units of density are kilograms per cubic-metre (kg/m^3) – the *rate* is mass per volume.

0.1.7 Force

Force is the subject of chapter 1. In that chapter we will take some time to carefully try to define what a force is. For us, now, we can just think of this as a push or a pull. Our job here is to explain what units we must use to measure a force.

I don’t doubt that you remember Newton’s 2nd Law: $\vec{F} = m\vec{a}$. Can we figure out the units on the right-hand side of this equation? Mass (kg) times acceleration (m/s^2) must equal force, and so must have the same units as force. The result, that force has units kg · m/s^2, will be a little awkward to have to keep writing out every time we need to specify a force. For this reason we define the unit

$$1\,\text{newton} = 1\,\text{kilogram} \cdot \text{metre}/\text{second}^2 \tag{44}$$

$$1\,\text{N} = 1\,\text{kg} \cdot \text{m/s}^2 \tag{45}$$

When a unit is named after a person it is convention to write the full name of the unit using lower-case, and the symbol for the unit as an upper-case. (Because "Newton" was the person and "newton" is the unit.)

The newton is an example of a *compound unit*; one that is composed of some products and/or ratios of the base Metric units. Other compound units that you will have seen in high school are the joule ($1\,\mathrm{J} = 1\,\mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s}^2$) which measures energy, and the pascal ($1\,\mathrm{Pa} = 1\,\mathrm{N/m}^2 = 1\,\mathrm{kg/m}\cdot\mathrm{s}^2$) which measures pressure. The purpose of defining a compound unit is simple: to write less. While the equivalent expression in the base units is important to be quantitatively correct, conceptually you should focus on associating a compound unit with the *type* of quantity it is used to measure. Knowing that a joule is $1\,\mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s}^2$ is usually not as important as looking at a quantity measured in joules and *thinking* about the concept of energy.

0.1.8 Other Units...

There are two other fundamental units of measurement that are required to quantify the physical phenomena related to energy and electricity. These are temperature, and electric charge. Temperature and thermal energy are, hopefully, familiar concepts. Electric charge is required to quantify the physics of electric current, electric potential, and electrical resistance. These units will be introduced, explained, and studied in chapters 4 and 6. We will leave them until then.

0.1.9 The Importance of Units

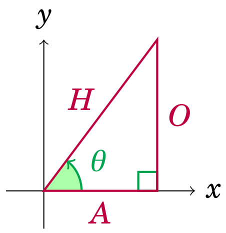
To reiterate what was stated at the beginning of this section: a quantity without units has no meaning. Including the units with a quantity gives it meaning, but also (as the preceding subsections have shown) provide critical information about what conversion factors, if any, need to included to interpret and use its value. Units determine how to interpret a quantity, physically. Units determine how to use a quantity, numerically. The proper inclusion and treatment of units will be strongly emphasized in this course.

0.2 Trigonometry

Trigonometry is the mathematics of triangles. It is expected that you have seen this in high school. We will quickly review this here since it is the building-block for the description of *vectors* (which are the subject of the next section in this chapter). We need to be able to describe vectors quantitatively since forces (chapter 1) and torques (chapter 2) are vectors, and those form the basis of *Biomechanics*.

While "trigonometry" is the proper name of this mathematical sub-discipline, we will refer to it as "trig" from now on.

0.2.1 Right-Angle Triangles



Of all the categories of triangle (like equilateral, isosceles, etc) the most useful type is the right-angle triangle, where one of the angles is 90°. The longest side, which is opposite the right-angle, is called the *hypotenuse*. The angle θ that specifies the direction of the hypotenuse is between the hypotenuse and what is called the *adjacent* side. The remaining side is called the *opposite*.

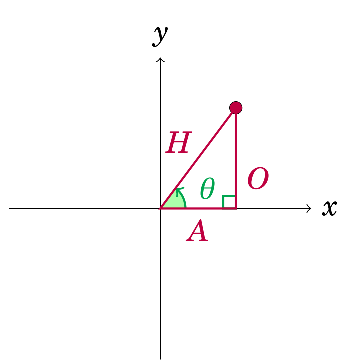
In the context of the physics of two-dimensional systems we usually make our measurements relative to a choice of Cartesian axes: the usual xy-axes. Placing the right-angle triangle with the adjacent side along the x-axis and the hypotenuse starting at the origin (as shown in the diagram) we measure the angle θ counter-clockwise from the $+x$-axis. The length A of the adjacent side is the x-coordinate, and the length O of the opposite side is the y-coordinate. The length H of the hypotenuse relates to the other two side by Pythagoras' Theorem:

$$H = \sqrt{A^2 + O^2} \tag{46}$$

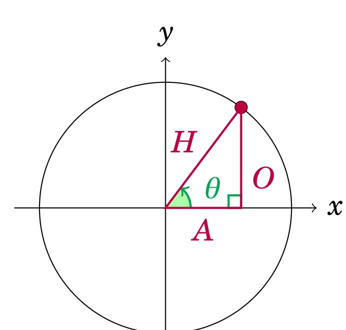
With the hypotenuse a fixed length the lengths of the other sides are determined by the angle. The relation between the angle and the lengths of the sides are given by the *trigonometric functions*.

0.2.2 Trig Functions

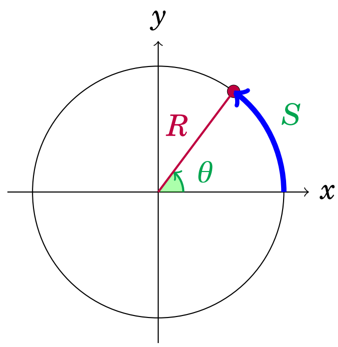
When the length of the hypotenuse and the angle of a right-angled triangle are given, the other two sides are determined. In the case of a fixed length of hypotenuse, the other two sides are *functions* of the angle. The function that gives the side opposite the angle in the *sine*, and the function that gives the side adjacent the angle is the *cosine*.



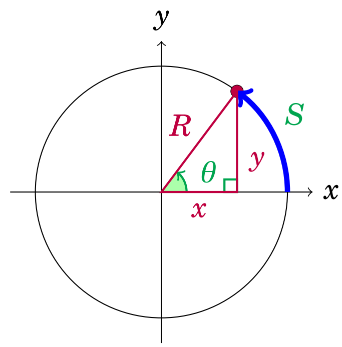
The triangle.



The circle specified by the triangle.



The point on the circle.
The angle is $\theta = S/R$.



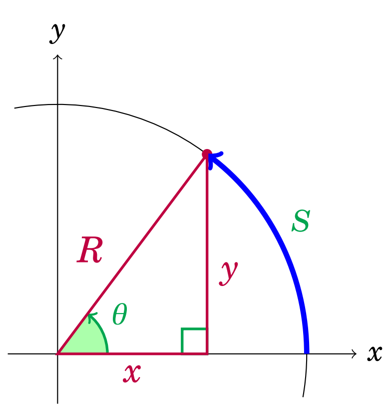
The triangle specified by
the point on the circle.

In the diagrams on the top, if we start with a right-angle triangle, we can form a circle with radius equal to the hypotenuse H. When this circle is centered on the corner of the triangle that has the angle θ, then the point on the circle where the triangle touches has coordinates $x = A$ and $y = O$.

In the diagrams on the bottom is the reverse approach, where starting from a point on the circle we can inscribe a right-angle triangle. This approach is important because it provides a definition of *angle*. In this construction the hypotenuse of the triangle is the radius. The point is located a distance S along the circumference from the x-axis. This defines the angle $\theta = S/R$ in terms of the measured lengths of the radius R and the arc length S.

The measurement of angle defined that way has units of *radians*. An angle that goes all the way around the circle has value equal to the circumference (the length of the arc) divided by the radius: $\theta = (2\pi R)\big/R = 2\pi\,\text{rad}$. Any other angle that is less than the whole circle will be a fraction of this quantity.

The more commonly used measure of angle is the *degree*. An angle that goes all the way around the circle is defined to be exactly 360°. Any other angle that is less than the whole circle will be a fraction of this quantity. From their definitions the conversion factor between these two units of angular measure is $\pi\,\text{rad} = 180°$. In practice and in this course we will measure angles related to geometry and vectors using degrees only. The only exception is when describing rotational motion and torque (chapter 2) where we will need to express angles in radians.



$x = R\cos\theta$
$y = R\sin\theta$

In a circle of radius R specifying an angle θ specifies an inscribed triangle. Consequently the sides of that triangle (the adjacent x and opposite y sides) are each a *function* of the radius and the chosen angle. These functions are called the ***cosine*** and ***sine***, and they relate to the triangle in this way:

$$x = R\cos\theta \tag{47}$$
$$y = R\sin\theta \tag{48}$$

These two trigonometric functions are fundamental in the sense that they can not be expressed in terms of other functions we already know, like the square-root. While these functions are defined in terms of angles measured in radians, our calculator will automatically convert the angles we specify in degrees to angles in radians before computing the function value. So we can talk about these trig functions in terms of angles in degrees.

Quadrants?

Initially the trig functions were defined using a triangle in the first quadrant with an angle between $0°$ and $90°$. But that is only one of the four quadrants. What about for triangles that are in one of the other quadrants? We need to have an answer to this since we will have forces that point towards the left (into the second or third quadrants), or downwards (into the third or fourth quadrants).

The trig functions work in ***all quadrants***. All that is required is that you measure the angle counter-clockwise from the $+x$-axis. Your calculator will give you the value of any of the trig functions for *any* value of angle. It is up to *you* to know which triangle you are using to the define the angle.

Domain of Trig Functions

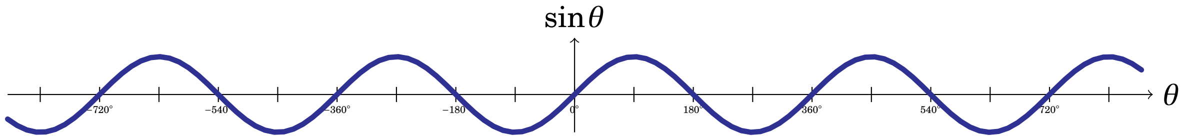
The first, most important, property of the trig functions is that they are *periodic*: they repeat. Specifically the value of $\sin(\theta)$ and the value of $\sin(\phi)$ are the same if θ and ϕ differ by an integer multiple of $360°$.

Positive angles are measured counter-clockwise around the circle, and *negative angles* are measured clockwise around the circle. Angles that are greater than $+360°$ and less than $-360°$ would correspond to having gone around the circle more than once. Such values of angle are necessary when describing angular or rotational motion. But for the geometry of triangles in the plane we only go around the circles once, at most, and we will use angles in the range $|\theta| < 360°$.

When solving geometric problems involving angles you can, because of the periodicity, always write your angles as positive quantities measured counter-clockwise from the $+x$-axis by adding an appropriate multiple of $360°$. But you calculator doesn't care: the domain of each of the trig functions is the entire real number line, and will calculate the correct value for any value of angle.

the Sine

The sine of an angle is the ratio of the opposite over the hypotenuse ($\sin\theta = O/H$). For our purposes in this course just think of it as the y-component of the triangle. As a function of the angle, its graph is like this:

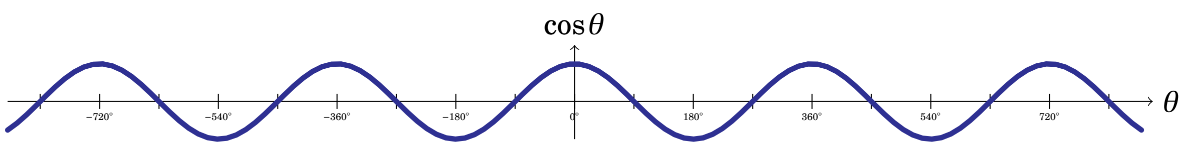


The function is zero at each and every multiple of 180°, positive and negative, including $\theta = 0°$. The maxima of this function are all $\sin\theta = +1$, and the minia are all $\sin\theta = -1$. The maxima (and minima) of the function are exactly mid-way between its neighboring zeros of the function. The zeros correspond to when the triangle has been flattened onto the x-axis. The maxima are when the triangle has been flattened against the $+y$-axis. The minima are when the triangle has been flattened against the $-y$-axis.

Note the spelling: "sine". This *sounds* like the word "sign", which would relate to whether a number is positive or negative. But it has nothing to do with that. The name "sine" is just a historical mistake (a mis-translation from the original Arabic) that we are stuck with. There is no deeper meaning to the name.

the Cosine

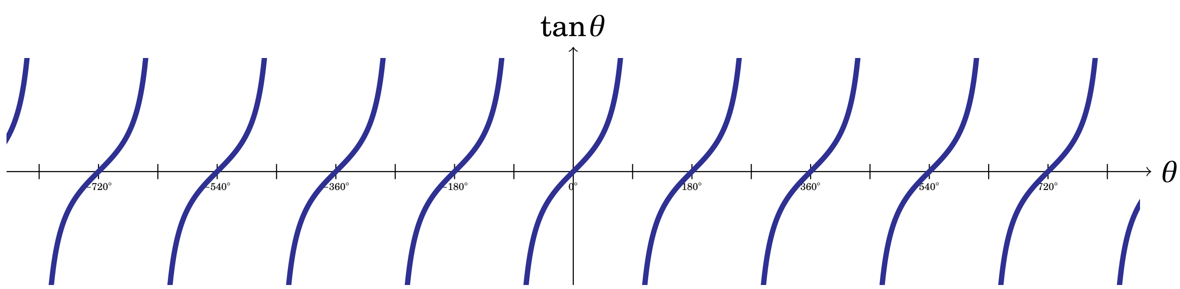
The cosine of an angle is the ratio of the adjacent over the hypotenuse ($\cos\theta = A/H$). For our purposes in this course just think of it as the x-component of the triangle. As a function of the angle, its graph is like this:



First notice that this function looks exactly like the sine function if it was shifted 90° to the left. (It is!) The maxima are all $\cos\theta = +1$, and the minima are all $\cos\theta = -1$, and those happen exactly half-way between two neighboring zeros. But now each of its zeros are at −90° plus an integer multiple of 180°. The "triangles" that correspond to these zeros are like those in the case of the sine, but with the role of the x and y axes exchanged.

the Tangent

The tangent of an angle is defined to be the ratio of the opposite over the adjacent ($\tan\theta = O/A$). Unlike the sine or cosine, where the denominator of the ratio was a positive constant (the hypotenuse), the value of the adjacent varies. Most significantly, when the triangle is collapsed onto the y-axis (for example when $\theta = +90°$ or $\theta = -90°$), the value of the adjacent goes to zero, and the ratio O/A diverges to plus or minus infinity ($\pm\infty$). This is its graph:



The tangent $\tan\theta = O/A$ is of use in those cases when we do not have (or do not need) the hypotenuse. If we have one side and the angle, the tangent lets us calculate the other side, without needing the value of the hypotenuse.

From its definition we find that the tangent function relates to the sine and cosine:

$$\tan\theta = \frac{O}{A} = \frac{(O/H)}{(A/H)} = \frac{\sin\theta}{\cos\theta} \tag{49}$$

0.2.3 Inverse Trig Functions

The trigonometric functions go from the angle to the ratio of sides. But, in many cases, we will have the sides of a triangle, but not the angle θ. In those cases, it is the *inverse* trigonometric functions go from the ratio of sides to the angle. Mathematically

$$\cos\theta = A/H \tag{50}$$

$$\theta = \cos^{-1}(A/H) \tag{51}$$

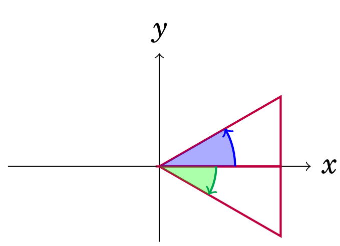
The cosine function is written "cos(·)", while the inverse cosine is written "$\cos^{-1}(\cdot)$", or sometimes as "arccos(·)".

When we use the inverse trig functions we must be cautious: one angle corresponds to one triangle, but there are two triangles that have the same ratio of sides. Our calculator will only give us one of the two possible answers. This is similar to when we use the square-root function, where our calculator tells us "$\sqrt{4} = 2$", when there are *two* answers ($\sqrt{4} = +2$ and $\sqrt{4} = -2$).

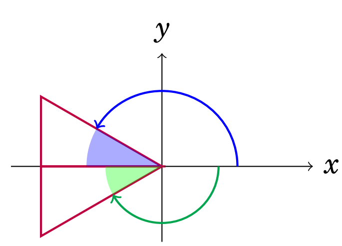
IMPORTANT : Your calculator is dumb

When you use any of the *inverse* trig functions on your calculator it will only be able to give an answer in two of the four quadrants. It is ***critical*** that you know what quadrant the answer should be in. The calculator will give you a number, but you will need to use your *brain* to interpret what that means about the actual angle.

the Inverse Cosine



These two triangles have
the same $\cos\theta = A/H$



These two triangles have
the same $\cos\theta = A/H$

In each of the diagrams above there is a pair of triangles, both with the same size of hypotenuse. Each in the pair is defined by an angle that is equal in magnitude, but opposite in sign, to the other triangle in the pair. In these pairs of triangles the adjacent side is common to both. For this reason the ratio A/H is the same for both triangles in the pair.

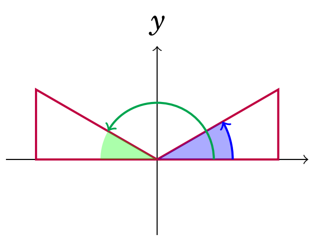
When we use our calculator to find the inverse cosine for these pairs of triangles

$$\theta = \cos^{-1}(A/H) \tag{52}$$

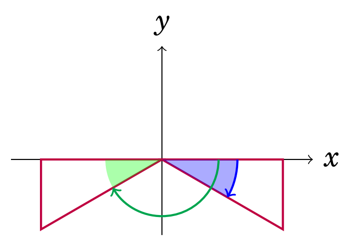
they have the same ratio A/H. So how will our calculator know which of the two triangles in the pair we are asking about? ***It can not***. The argument "A/H" of the function is a single number; it does not have information about the sign of A or H separately, but only their ratio. For the angles shown in the diagrams above for the inverse cosine (and below for the inverse sine and tangent) the value that your calculator will return is the one coloured in blue. If you are trying to find the angle for the triangle coloured in green, then you will have to use your brain to "convert" the answer into the correct quadrant. (Just like you would have to choose "$\sqrt{4} = -2$".)

the Inverse Sine

For the inverse sine our calculator will tell us the angle corresponding to a given ratio of O/H. The diagram below shows the pairs of triangles that have equal values of O/H. Angles noted by the blue wedge and the green wedge have the same size, but the angle for the green triangle is given by the green arrow, measured from the $+x$-axis.

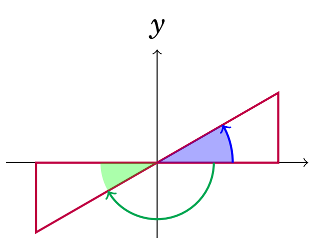


These two triangles have the same $\sin\theta = O/H$

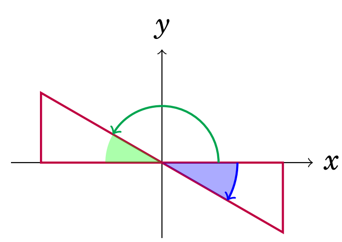


These two triangles have the same $\sin\theta = O/H$

the Inverse Tangent



These two triangles have the same $\tan\theta = O/A$

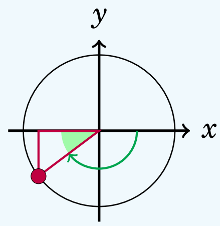


These two triangles have the same $\tan\theta = O/A$

Example 0.4 : Quadrant?!

What is the angle between the positive x-axis and the line from the origin to the point $x = -4\,\text{cm}$, $y = -3\,\text{cm}$?

When solving problems of trigonometry ***always*** draw the triangle. It is very tempting to simply "jam numbers into your calculator", but in the case of inverse trig functions that approach will be *wrong 50% of the time*.



From the text description of the problem, the point is in the third quadrant. With the hypotenuse connecting the origin with the point, and the adjacent side on the x-axis, the triangle is also in the third quadrant. From this we see that the angle must be in the range $-180^\circ < \theta < -90^\circ$ when measured clockwise from the $+x$-axis. (If the angle is measured counter-clockwise, then $+180^\circ < \theta < +270^\circ$. Both answers are equivalent.)

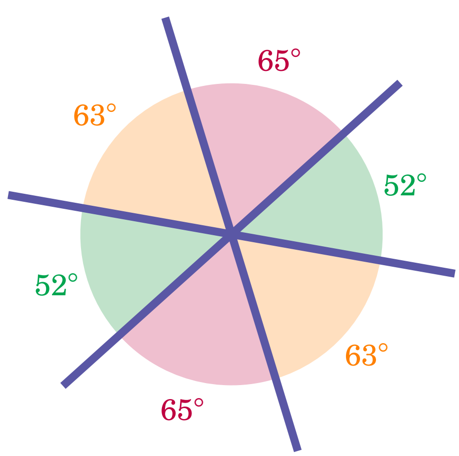
With the adjacent and opposite sides of the triangle specified the value of the tangent function is their ratio: $\tan\theta = O/A = (-3\,\text{cm})\big/(-4\,\text{cm}) = +3/4 = +0.75$. Our calculator will tell us (rounding to the nearest integer) that $\tan^{-1}(0.75) = 37^\circ$; but that is in the *first* quadrant, corresponding to a triangle whose adjacent and opposite sides are both *positive*. To translate this result to a triangle in the third quadrant, with adjacent and opposite sides both *negative*, we need to subtract (or add) 180°. This gives

$$\theta = 37^\circ - 180^\circ = -143^\circ \tag{53}$$

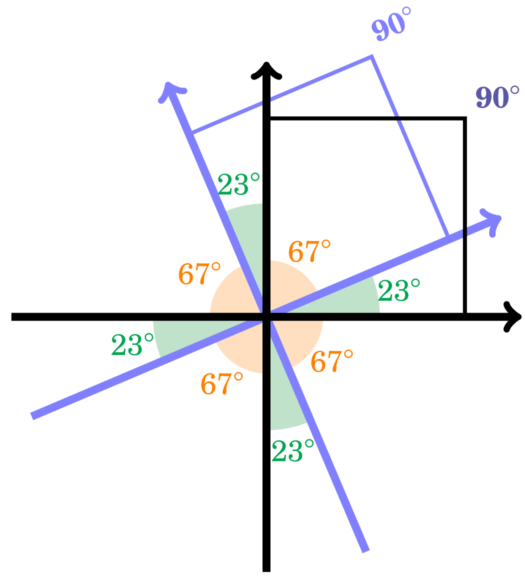
(or equivalently +216° by adding, if measure the angle counter-clockwise). The result matches our expectation which was set by our drawing of the triangle.

0.2.4 Some basic geometry

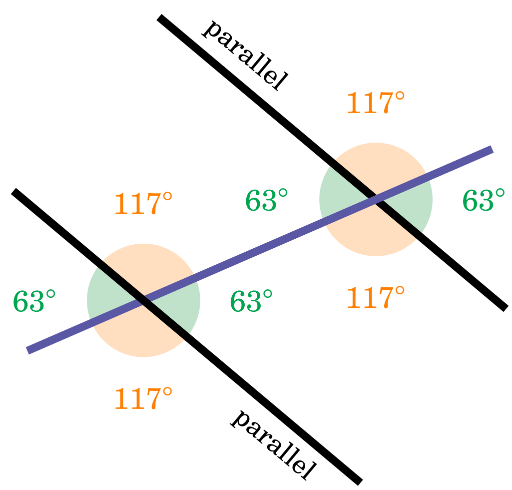
When lines intersect the angles formed relate to each other. Below is an example of three lines passing through a common point. Note how the angles opposite each other across the point of intersection are equal.



Another common geometry that we will encounter is when we have a set of coordinate axes that are rotated by a specific angle. In the diagram below note how the angles formed between the standard (horizontal an vertical) axes and the rotated axes all relate to each other.



This geometry occurs when we consider objects on an inclined surface. There are the axes corresponding to horizontal and vertical, and there are axes that are parallel and perpendicular to the surface, which is tilted.



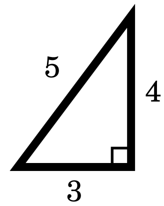
In the case where a line crosses two parallel lines the angles formed relate as shown in the diagram above. This is know as the "transverse-parallel theorem" (or the "Z-theorem").

0.3 Vectors

Describing how something is moving can not be done with a single number. Consider traveling in a car at 100 km/h. Traveling along the highway at that speed might be the start of a fun trip. Traveling at that speed towards the ditch would . . . not be fun. The speed by itself is not enough to specify movement, direction is also necessary. But movement is a single concept, and the mathematical object that describes it is a ***vector***.

0.3.1 Vector $\neq$ Number

A vector is not a number. To understand what that means have a look at this familiar triangle, below:



You know this:

$$\mathbf{5} \neq \mathbf{3} + \mathbf{4} \tag{54}$$

So how does the side of length 3 "add" to the side of length 4 to give the side of length 5? You can see from the diagram that the ***direction*** of the sides are important. This point of this? That a single number (the length) is not enough to specify a vector. A vector is not a number.

A vector may not be a number but it can be described and quantified using *numbers* – with an emphasis on there being more than one number required.

0.3.2 Vector Notation

When working with vectors there are three similar-looking symbols that have very different meanings. At each step you must be very clear which you are actually working with:

$\vec{C}$ The letter gives the vector a name, and we read this as "the vector C". Look very closely at the symbol and notice the arrow above the symbol. It is that arrow that labels the quantity as a vector. This is a vector and you have to remember that it is not a number.

C_x The letter "C" tells us that we are talking about something *related* to the vector $\vec{C}$. This number is called a ***component*** of the vector. The sub-script "x" tells us that this quantity is the x-component of the vector $\vec{C}$. This quantity is a number, and its sign (positive or negative) is important. But the sign of C_x is not by itself the "direction" of the vector $\vec{C}$. You need both components to determine the direction of the vector: You need the x-component to specify how much the vector points to the right or to the left; and You need the y-component to specify how much the vector points upwards or downwards.

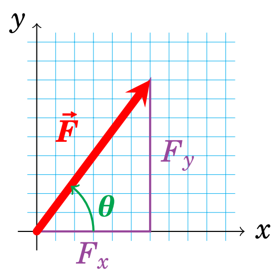
C The letter C written without an arrow above, and without a sub-script below, is the magnitude of the vector $\vec{C}$. The magnitude C is a non-negative number. It is the "length" of the hypotenuse of the triangle formed by the vector and the xy-axes. It relates to the components of the vector by Pythagoras' Theorem:

$$C = \sqrt{C_x^2 + C_y^2} \tag{55}$$

In the context of forces, it is the answer to the question "how hard are we pushing?" The magnitude does not have any information about the direction of the vector – have another look at the 3-4-5 triangle in the previous sub-section to understand this.

0.3.3 Vector Components

The usual description of a vector is a quantity "with magnitude and direction". The magnitude is a number (with units), and the "direction" is an angle (also a number) measured from a specified axis. An alternative – and much more useful – way to describe a vector using numbers is to specify its *components*. These two numbers answer "how much to the left or right?" ***and*** "how much up or down?"



Constructing a right-angle triangle that has the vector as its hypotenuse, the components are the adjacent side (F_x) and the opposite side (F_y).

$$F_x = F\cos\theta \tag{56}$$

$$F_y = F\sin\theta \tag{57}$$

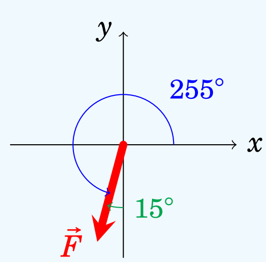
As long as you measure the angle θ counter-clockwise from the $+x$-axis these two equations will give the correct sign for each component, regardless of what quadrant the vector points into.

In most cases you will not be given a diagram of the force, and you will be "using numbers to find other numbers". It is critical to recall how dumb your calculator is. (As we saw in subsection 0.2.3 regarding the inverse trig functions.) ***Always*** draw, or qualitatively sketch, the vector so that you know the quadrant. This is necessary to get the signs of the components correct, and/or to get the angle into the correct quadrant. Always sketch the vector.

Example 0.5 : Vector components

A force of magnitude 42.1 N is pushing downwards, 15° to the left of the vertical. What are its components?

We always sketch the vector before calculating anything to help guide our process. It will also provide an idea of what the answer will be so that we can check our result at the end.



From the diagram we see that both components will be negative ($F_x < 0$ N and $F_y < 0$ N) since it points into the third quadrant, and that the y-component will be larger than the x-component ($|F_y| > |F_x|$) since it is closer to the y-axis than the x-axis. We will use this information to check our result at the end.

The angle we were give was measured from the $-y$-axis (at $3 \times 90° = 270°$). To calculate the components we need the angle measured counter-clockwise from the $+x$-axis.

This is $\theta = 270° - 15° = 255°$ (the angle noted in blue in the diagram).

$$F_x = F\cos\theta = 42.1\,\text{N}\cos 255° = -10.9\,\text{N} \tag{58}$$
$$F_y = F\sin\theta = 42.1\,\sin 255° = -40.7\,\text{N} \tag{59}$$

Checking with our qualitative result above we find that this agrees: both components are negative, and the y-component is largest.

0.3.4 Adding Vectors

Experience and experiment teaches us that forces combine. When two people push on an object its motion is affected as if there was a single force acting on it. That single effective force is the *sum* of the applied forces.

Adding Two Vectors

The rule for summing (adding) two vectors is the *parallelogram law*:

This diagram is meant to show that the sum of the two vectors can be found by moving along the direction of one vector, and then moving along the direction of the other vector. The sum then points from when you started to the new place where you arrived. This also shows how the order does not matter ($\vec{A} + \vec{B} = \vec{B} + \vec{A}$) and that you get the same result from adding in any order.

Since A_x measures the horizontal component of $\vec{A}$ and B_x measures the horizontal component of $\vec{B}$, moving along $\vec{A}$ and then along $\vec{B}$ will move you horizontally an amount $A_x + B_x$, and this must be C_x. For this reason the sum of vectors can be calculated using their components like this:

$$C_x = A_x + B_x \tag{60}$$
$$C_y = A_y + B_y \tag{61}$$

In cases where we add more than two vectors, the x-component of the result is the sum of all the x-components, and the y-component of the result is the sum of all the y-components.

Adding More than Two Vectors

Example 0.6 : Adding vectors

Three forces in the xy-plane are being exerted on an object: $\vec{A}$ of magnitude 4.47 N pointed 66° below the $+x$-axis; $\vec{B}$ with x-component -2.00 N at 30° to the left of the $+y$-

axis; and $\vec{C}$ of magnitude 1.37 N at 45°. What is the magnitude and direction of $\vec{D}$ the sum of these forces? (There is no significance to any of the letters used to name these forces.)

When presented with a problem of vectors the first thing you must do is *sketch the vectors*. When finding the components of a vector from its magnitude and direction, the sketch will give you information about what the signs of the components will be, as well as their size relative to each other.

Stop! Sketch them now. My sketches will be below.

The sum of forces is

$$\vec{D} = \vec{A} + \vec{B} + \vec{C} \tag{62}$$

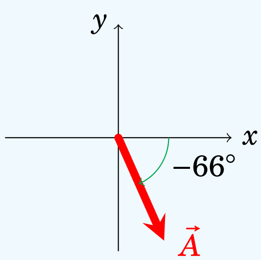
In terms of the components, this means

$$D_x = A_x + B_x + C_x \tag{63}$$

$$D_y = A_y + B_y + C_y \tag{64}$$

To calculate these two equations we will first need the x and y components of the three forces. The components of $\vec{A}$ and $\vec{C}$ are easy to find, since we are given a magnitude and the direction for each.

The sketch of $\vec{A}$ is:



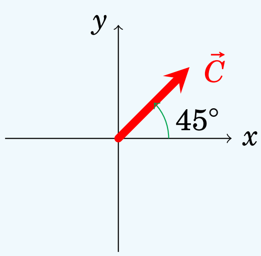
The sketch shows us to expect that $\vec{A}$ points into the fourth quadrant ($A_x > 0\,\mathrm{N}$ and $A_y < 0\,\mathrm{N}$) and that $|A_x| < |A_y|$. Calculating the components of $\vec{A}$ we find

$$A_x = 4.47\,\mathrm{N}\cos(-66°) = +1.818\,\mathrm{N} \tag{65}$$

$$A_y = 4.47\,\mathrm{N}\sin(-66°) = -4.084\,\mathrm{N} \tag{66}$$

which matches our expectations. (If we respect significant figures, we would only keep two decimal places – but we will wait until our final result before we apply rounding.)

The sketch of $\vec{C}$ is:

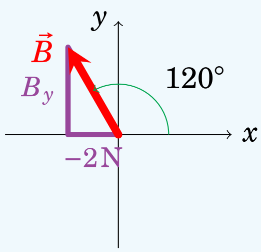


The sketch shows us to expect that $\vec{C}$ points into the first quadrant ($C_x > 0\,\mathrm{N}$ and $C_y > 0\,\mathrm{N}$) and, because the angle is 45°, that $|C_x| = |C_y|$. Calculating the components of $\vec{C}$ we find

$$C_x = 1.37\,\mathrm{N}\cos(45°) = +0.969\,\mathrm{N} \tag{67}$$
$$C_y = 1.37\,\mathrm{N}\sin(45°) = +0.969\,\mathrm{N} \tag{68}$$

which matches our expectations.

The vector $\vec{B}$ is specified by a component ($B_x = -2.00\,\mathrm{N}$) and a direction (30° to the left of the $+y$-axis, which is 120° from the $+x$-axis). When we sketch it, we see that this describes a triangle in the second quadrant.

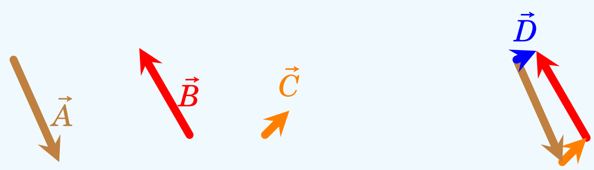


The sketch shows us to expect that $\vec{B}$ points into the second quadrant (since $\theta = 120°$, so that $B_x < 0\,\mathrm{N}$ and $B_y > 0\,\mathrm{N}$) and, because it is closer to the $+y$-axis than the $-x$-axis, that $|B_x| < |B_y|$. Since $\tan\theta = B_y/B_x$ we can use $B_y = B_x \tan\theta$ to find its y-component. Calculating the components of $\vec{B}$ we find

$$B_x = -2.00\,\mathrm{N} \tag{69}$$
$$B_y = B_x \tan\theta = -2.00\,\mathrm{N}\tan(120°) = +3.464\,\mathrm{N} \tag{70}$$

which matches our expectations.

Before we calculate the sum of the vectors numerically, it is a good idea to sketch the vectors to get a qualitative idea of what the result will be. (The different colours of the vectors are just so we can distinguish them from each other, and have no other significance.)



Since we have the numerical values of all the components of the vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$, we can calculate their sum:

$$D_x = A_x + B_x + C_x \qquad D_y = A_y + B_y + C_y \tag{71}$$
$$= (+1.818\,\mathrm{N}) + (+0.969\,\mathrm{N}) + (-2.00\,\mathrm{N}) \qquad = (-4.084\,\mathrm{N}) + (+0.969\,\mathrm{N}) + (+3.464\,\mathrm{N}) \tag{72}$$
$$= +0.79\,\mathrm{N} \qquad = +0.35\,\mathrm{N} \tag{73}$$

The Zero Vector

When adding vectors, specifically forces, there is a special result that we will often require: that the vectors *cancel* each other. If it is just two vectors, and one goes forwards, the other comes back the same amount, and you are back where you started. If three vectors are being added, then (graphically) their addition forms a triangle, where the start and finish are at the same place.

IMPORTANT : The Zero Vector

A vector is not a number. But there is an important vector that ***looks*** like it is an exception: the zero vector, $\vec{0}$. This is the vector whose magnitude is zero. (Because of this, the zero vector does not have a direction.) For historical reasons we are stuck with writing the zero vector as $\vec{0}$, even though it is not a number. One way to rationalize this is to see that each component of this vector is zero.

Example 0.7 : Vectors that sum to zero

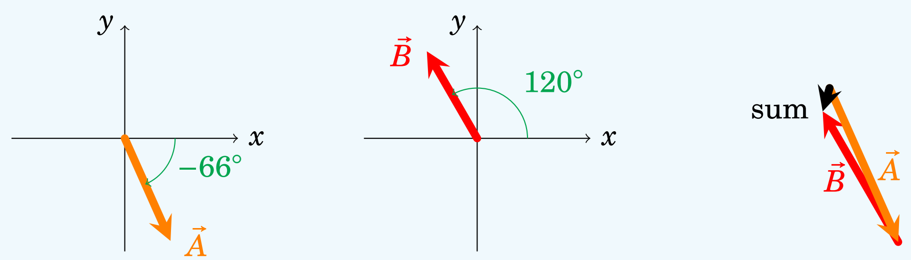
Three forces in the xy-plane are being exerted on an object: $\vec{A}$ of magnitude 4.47 N pointed 66° below the $+x$-axis; $\vec{B}$ with x-component -2.00 N at 30° to the left of the $+y$-axis; and $\vec{C}$ whose magnitude and direction are unknown. What is the unknown force $\vec{C}$ if these three forces sum to zero?

The equation that defines the situation is

$$\vec{0}\,\mathrm{N} = \vec{A} + \vec{B} + \vec{C} \tag{74}$$

When we write out the x and y components of this equation we will have two equations for the two unknowns: C_x and C_y. Since the components are all numbers, the terms in each of the equations will obey all the usual rules of algebra, and will have values we can find with our calculator.

The forces $\vec{A}$ and $\vec{B}$ were already used in the previous example (0.3.4). Using what we learned in that example we can *qualitatively* estimate what the force $\vec{C}$ must look like. The forces $\vec{A}$ and $\vec{B}$ look like, and sum to this:



Both forces are roughly the same magnitude, but point in slightly different directions. The force $\vec{B}$ is 30° from the vertical axis, while the force $\vec{A}$ is only 24° from the vertical axis. For the sum of all three forces to be zero the force $\vec{C}$ must be the ***opposite*** of $\vec{A}+\vec{B}$. Qualitatively, from the diagram we have drawn, we can see that $\vec{C}$ will be small in comparison to the other two, and that will point slightly *upwards*. This result is not the conclusion of the problem, but gives us an estimate against which we can check our quantitatively calculated answer below.

The components of the forces $\vec{A}$ and $\vec{B}$ were already calculated in the previous example (0.3.4):

$$A_x = +1.82\,\text{N} \qquad B_x = -2.00\,\text{N} \tag{75}$$

$$A_y = -4.08\,\text{N} \qquad B_y = +3.46\,\text{N} \tag{76}$$

Using these values we find that the x and y components of the equation for the sum of all three vectors are:

$$0\,\text{N} = A_x + B_x + C_x \qquad 0\,\text{N} = A_y + B_y + C_y \tag{77}$$

$$0\,\text{N} = (+1.82\,\text{N}) + (-2.00\,\text{N}) + C_x \qquad 0\,\text{N} = (-4.08\,\text{N}) + (+3.46\,\text{N}) + C_y \tag{78}$$

$$C_x = +0.18\,\text{N} \qquad C_y = +0.62\,\text{N} \tag{79}$$

As found qualitatively this vector is small in comparison to the other two, and points mostly upwards ($C_y > 0\,\text{N}$ and $|C_y| > |C_x|$).

0.4 Logarithms

In chapter 5 when we will be studying the energy carried by sound waves we will need to use the *logarithm* function. You are supposed have seen this briefly in high school, but it's understandable if you don't recall this topic. This section is meant as a review of (introduction to?) the logarithm function. This review will be most effective if you get your calculator out and follow along with the examples. So get your calculator!

The idea of the logarithm is that it is the inverse of the exponential. To warm up to this idea let's review an inverse you know: the square-root.

0.4.1 Remember the Square-Root?

The *square* of a number is that number multiplied by itself: $y = x^2 = x \times x$. It is a mathematical fact that if you choose a non-negative number y, there is a number x such that $y = x^2$. Both of these operations are functions (they map each number onto another single number) and each is the *inverse* of the other.

$$y = \text{square of } x \qquad y = x^2 \tag{80}$$

$$x = \text{square root of } y \qquad x = \sqrt{y} \tag{81}$$

When you have an equation involving a square, you can usually solve it using a square-root. For example, if you are told that $x^2 = 16$, you know that $x = \pm 4$. But if $x^2 = 3.7$, you must calculuate $x = \pm\sqrt{3.7}$. That comes from applying the square-root to both sides:

$$x^2 = 3.7 \tag{82}$$

$$\sqrt{x^2} = \sqrt{3.7} \tag{83}$$

$$x = \sqrt{3.7} = 1.923\ldots \tag{84}$$

The $\pm$ we add by hand since we know that there are two consistent solutions to the original equation $x^2 = 3.7$ ($(+1.923)^2 = 3.7$ and $(-1.923)^2 = 3.7$) but our calculator gives only the positive root. The relationships $\sqrt{x^2} = x$ and $(\sqrt{x})^2 = x$ are true since the square-root is the *inverse* of the square.

0.4.2 Powers of Ten

The square of a number is that number multiplied by itself. Think of it this way: take *two* copies of x and then multiply them together. The idea of an *exponent* is to take a specific number of copies (not just two) and then multiply them together. For example, if the exponent is five, then

$$y = x^5 = x \times x \times x \times x \times x \tag{85}$$

In that example the number "5" is called the exponent and the quantity "x" is called the *base*. As written $y = x^5$ is a function that maps values of x onto y, and x is the independent variable. What we are going to do next is look at cases when the base is a fixed number and the *exponent* is the variable.

Ten raised to the power five is

$$10^5 = 10 \times 10 \times 10 \times 10 \times 10 = 100\,000 \tag{86}$$

In general, ten raised to an positive integer n is a one followed by n zeros; the exponent tells us how many times the decimal place has been "moved over" towards the right.

Algebraic rules of exponents:

$$10^a \times 10^b = 10^{(a+b)} \tag{87}$$

$$\left(10^a\right)^b = 10^{(a \times b)} \tag{88}$$

There is an important variant of the first property, found when we take a *ratio* of two exponentials:

$$\frac{10^a}{10^b} = 10^{(a-b)} \tag{89}$$

▲FIX: To be written. The exponent zero. Negative exponents. Non-integer exponents. Graph of $\log(x)$. Comment on steepness.

0.4.3 The Logarithm

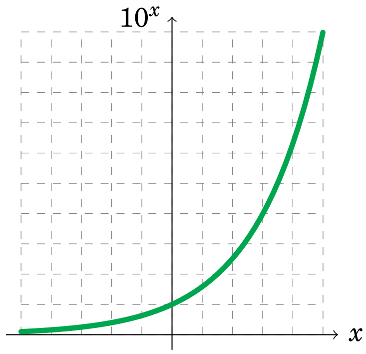
With the idea of the function $y = 10^x$ comes the idea of its inverse:

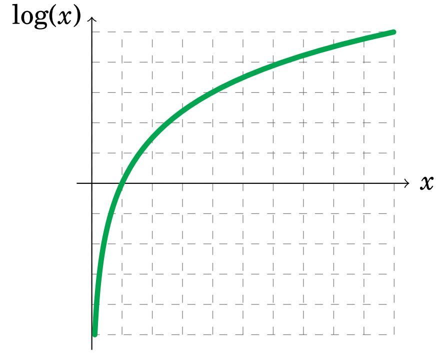
$$y = 10 \text{ to the power } x \qquad y = 10^x \tag{90}$$

$$x = \text{logarithm of } y \qquad x = \log(y) \tag{91}$$

Your calculator may have *two* different logarithm buttons on it. One will be labeled `LOG`, and that's the one that we want. (If there is another one, it might be labeled `LN`. Don't use the `LN` function!)

Graph of 10^x. Graph of $\log(x)$. Comment on their steepness, and their asymptotes.





You know that if you square ten you will get one hundred. You know that if you take the square-root of one hundred you will get ten. The logarithm is the function that tells you that *two* is the power of ten that equals one hundred.

$$10^2 = 100 \tag{92}$$
$$2 = \log(100) \tag{93}$$

This pattern is true for any exponent. For example, when the exponent is seven:

$$10^7 = 10\,000\,000 \tag{94}$$
$$7 = \log(10\,000\,000) \tag{95}$$

And when the exponent is negative three:

$$10^{-3} = 1/1000 = 0.001 \tag{96}$$
$$-3 = \log(1/1000) \tag{97}$$

Essentially the logarithm tells you where the leading digit is relative to the decimal place. Notice also that there is no value of x for which 10^x will be less than zero. For this reason the logarithm is not defined for negative arguments (just like how $\sqrt{-1}$ is not defined).

Now, try these in your calculator:

$$10^{5.81} = 645\,654.229\ldots \tag{98}$$
$$5.81 = \log(645\,654.229\ldots) \tag{99}$$

Note that the number 645 654.229 is between $100\,000 = 10^5$ and $1\,000\,000 = 10^6$, so its logarithm is between 5 and 6. As with the square-root the exponential (base-10) and the logarithm are defined for non-integer values. Just remember that, since $10^x > 0$ for *any* value of x, the logarithm $\log(y)$ is only defined for $y > 0$.

When you have an equation involving a square, you can usually solve it using a square-root. Similarly if you have an equation involving the unknown as the exponent of 10, you can usually solve it using the logarithm. For example, if $3.7 = 10^x$, then $x = \log(3.7)$ since the exponential base-10 (the function 10^x) and the logarithm are inverses of each other. Similar to the relation between the square and the square-root, the relations $\log(10^x) = x$ and $10^{\log(x)} = x$ are true between the exponential and the logarithm. These relationships will be explored in the Exercises.

0.5 Mathematical Symbols

Another title for this section might be "what's with all these weird letters?"

0.5.1 The Utility of Symbols

One way to characterize physics is to say that it studies the quantitative relationships between measurable quantities. An example relationship would be "the sum of forces acting on an object equals its mass times its acceleration". This expression can, of course, be refined and nuanced with further words to clarify, sharpen and deepen its meaning and utility.

But if you have gathered data and need to use this relationship to calculate predictions, you don't want to have to spend your time writing out this long sentence over and over. It's much faster fast to write "$\sum \vec{F} = m\vec{a}$".

If we need to document or communicate our calculations and results through writing, then we can choose a form of writing that suits our purpose.

0.5.2 The Range of Symbols

If we limit ourselves to letters from the beginning of the word that names a quantity then we run into a problem: there are multiple quantities and concepts that begin with the same letter. In some cases it is a problem we are stuck with, and can not change, because the choice of letter has been fixed by historical convention. For example:

- The upright letter "m" is the symbol for the base unit of distance, the metre, as well as the prefix "milli" (10^{-3}), while the upper-case "M" is the prefix "mega" (10^{+6}). But the italicized letter "m" or "M" is conventionally used to denote the mass of an object.
- $\vec{T}$ is the force of tension exerted by a rope (sub-section **??**), T is the thermodynamic temperature (section 4.2), and T is the time between repetitions of oscillation (section 5.4).
- In chapter 4 we will talk about how K is the kinetic energy of an object, kelvin K is the unit of temperature, and $\mathscr{K}$ is the thermal conductivity of a material. In section 3.3 we will remind ourselves of how k is the spring constant of an object defined through Hooke's Law.
- The symbol "C°" is the unit of temperature for the Celsius scale, while "C" the coulomb is the unit of electric charge. The symbol "$\mathscr{C}$" denotes heat capacity (each type of material has its own value of this physical property), while "c" denotes the speed of light in vacuum (which has the exact value 299 792 458 m/s).

▲FIX: To be written.

The Latin alphabet "ABC…" and "abc…". Reserved symbols and conventional uses (like "F" for "force", and "x" for the horizontal axis). Running out of symbols.

The Greek alphabet "$\alpha\beta\gamma\delta\epsilon\ldots$". You already know one of them: "θ" which is traditionally used to denote an angle.

0.5.3 Common Greek Symbols

There are only twenty-six letters in the English alphabet – far too few for enormous number of objective quantities that are ruled by physical laws.

The Greek letter theta "θ" which is traditionally used to denote an angle.

The summation $\sum$ is the upper-case version of the Greek letter sigma.

Torque (the topic of chapter 2), which is the rotational counterpart of force, is symbolized by "τ" the Greek letter tau.

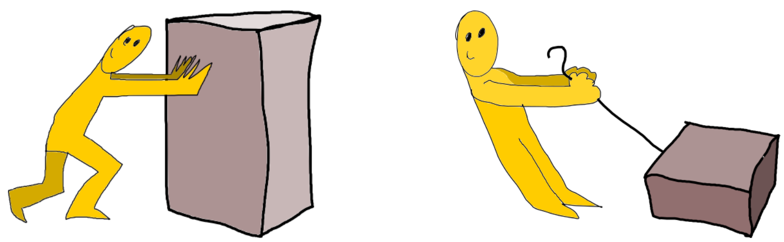
Noting a difference or change Δ is the upper-case version of the Greek letter delta.

In chapter 3, when we study the properties of materials, we will use the Greek letters sigma σ and epsilon ϵ.

In chapter 5, when we study waves, we will use the Greek letter lambda λ.

Chapter 1

Forces



For us, a force is a push or a pull. In the context of the Physiotherapy Technology program the forces we need to think about are those exerted by or exerted on the human body. As a result of those pushes and pulls there will be forces generated *inside* the body, between the muscles, tendons, ligaments, cartilage and bones. Being able to think objectively about these concepts will form the foundation for your *Biomechanics* and *Physiology* courses.

1.1 What is a Force?

A technically precise definition of a force would be "an interaction that could change an object's motion". For us, to say that a force is a push or a pull will be a good start. In this chapter we will now try to refine that idea, and be more rigorous in how we think about forces. Conceptually we will need to be careful and precise in how we speak about forces. Physically we need to see a force as an interaction that could change an object's motion. Quantitatively we need to be able to work with forces as vectors.

1.1.1 "The Object"

When speaking of forces we must be clear to identify what the force is affecting. There will always be that which acts, and that which is acted upon. The action itself is the force. Here are some examples:

"Gravity pulls down on the apple."

"The racket hits the tennis ball."

"The woman pushes the battery into its socket."

"The child is dragging their toy across the lawn."

In all those examples the sentence are structured as "A acts on B". The thing "A" is the cause of the force and "B", the thing that feels the force, is *the object*. When speaking of, thinking about, or analyzing the effect of force it is critical to *identify the object*. This is so important that it will be considered *Step 0*.

Pictured is a person pressing down on a scale (a device with a spring inside whose deformation measures the weight on it). Which way is “the force”? The answer depends upon the choice of object. If the object is the scale, then the force acting on the scale is downwards. If the object is the person, then the force acting on the person is upwards. These two forces are related (by Newton’s 3rd Law of “action and reaction”), but they are separate forces acting on separate objects. If we need to talk about a force we must specify what it is acting upon.

1.1.2 Thinking about Interactions

There are some important words to notice when speaking of forces. If we’re pushing or pulling on an object, and the cause of the interaction is the subject of the sentence, then we say that we are *applying* a force to the object. We may also say that we are *exerting* a force on the object. Using the word “exert” is intended to make you think about the related effort, to help place your imagination in the context with you being the cause of the interaction. (The idea of effort returns in chapter 4 when we study *energy*.) When talking about forces always make an effort to be aware of what two things are interacting, and which one is the object on which the force is acting. How we speak models how we think. And being mindful of how we speak can change how we think.

Gravity is the interaction between the Earth and the masses of objects that are near it, relative to the size of the Earth. In this course the context will always be on or near the surface of the Earth. Therefore gravity will always be present in the context of this course. For an object near the Earth the force of gravity will be downwards, and will have a magnitude equal to the object’s mass times $g = 9.81$ N/kg. All exercises and problems that investigate the effect of the sum of all forces acting on an object will include gravity.

Do not be afraid to draw pictures of what is happening. If it helps, try building a little model of what is happening. Forces (and later in the course, Torques) are all about how things relate to each other geometrically. Use any method that works for you to help yourself understand the geometry of the situation.

1.1.3 Forces are Vectors

When we push on an object there are three elements: where on the object we are pushing, how hard we are pushing, and the direction we are pushing.

The importance and consequence of *where* on the object we are pushing is the subject of chapter 2. We will leave this for now.

How hard we are pushing is quantity we would measure in newtons. The direction we are pushing is the angle we would measure in degrees relative to a reference direction. For this reason, force itself, as a thing, is not a *single* number. To quantitatively describe a force we need at least two numbers: the magnitude and the direction. This categorizes force as a vector. (This means that, alternatively, we could also quantitatively describe a force using its components.)

Facts like where on the object we are pushing and the direction we are pushing are geometric in their nature. Vectors we draw as arrows, and force is a vector quantity. Because of this it is reasonable to ask: How can we draw "a force acting on an object"?

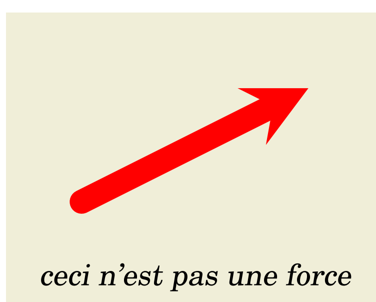
1.1.4 How to Draw a Force?

Do me a favour: do a web search for the painting by the French surrealist Magritte with the words "*ceci n'est pas une pipe*". The painting, titled "*La Trahison des images*", is both a joke and a serious statement. The joke is that a painting of a thing is not the thing itself, obviously. The serious part is that we should be careful to not think that a representation of something is the same as the thing itself.

I bring in this abstract philosophical point to focus us on an important question: If we are going to draw a force, then what do we want or need that representation to convey? Think about the goal. Interaction is determined by geometry, and thinking about geometry is always aided by drawing it. Being able to include forces in our drawings will help us reason about them. The properties of force that are important were listed in the previous sub-section: where on the object we are pushing, how hard we are pushing, and the direction we are pushing.

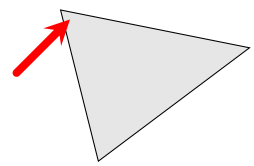
In general when we draw it is with a collection of lines of certain lengths and directions on the page. We will make a representation of a force by drawing a specific collection of lines: an arrow.

We begin with a line. The direction of the line is to be parallel to the force. The length of the line will be, in some way, proportional to the magnitude of the force. To distinguish the direction of a force we decorate one end of the line with an "arrow head". The direction of the force is from the un-decorated end towards the arrow-head end.

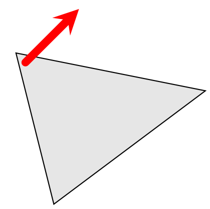


It may not be "a thousand words", but I can draw three or four arrows much, much faster than I can type the sentences that would convey the same information about the forces acting on my body as I sit typing these words.

If we have drawn a picture of the object we can place the arrow of the force on the object to indicate where on the object it is acting. Although there is one unresolved question: Do we place the tip or the tail at the position on the object where the force is acting?



A Push



A Pull

In an idealized situation it is acceptable to think that pushing towards the right will achieve the same effect as pulling towards the right. However, in reality you know that someone pulling on your right arm is different from someone pushing on your left shoulder. In this text I will choose to draw force with the arrow head tip touching the object if it is a force that is pushing on the object. If the force is, instead, pulling I will draw it with its tail on the object with the arrow head pointed away.

There is no real significance to the colour used to draw the forces. In this text I will usually use red to draw force vectors. If there are multiple forces, and I need to draw them very close to each other, or even overlapping, I may use shades of orange, red and brown, so that we can visually distinguish between the different forces. There is no significance to these colour schemes; it's just for clarity.

1.2 The Types of Forces

There are two categories of force.

The first category of forces are those exerted when two objects touch each other. A book resting on a table, or ladder leaning against a wall, experience these kinds of forces. There are also forces exerted between the pieces of an object itself because of how they are connected to each other. The fibers inside a rope that is being pulled, or the parts of a door as you push it closed, experience these kinds of forces. All of these belong to the category of *contact* forces.

As you might have guessed, the other category are *non-contact* forces. Gravity is the ever-present example of this type of force. You do not have to touch the Earth to feel the effect of gravity. In fact I would argue that when are not touching the Earth is when you become most aware of gravity! This fact is part of your foundation of how you think about the world, and you are always aware of it, even if not intellectually or consciously. In what follows we will try to become more rigorous in our thinking about gravity.

Later in the course (chapter 6) we will study the effects of another non-contact force: Electrical force. Electrical forces are present all about us, but we are almost never aware of them. Their indirect effects, through lighting, appliances, and computers, form the basis of our day-to-day lives. But direct electrical forces are almost never seen. The reason why will be uncovered in chapter 6.

1.2.1 Contact

When two objects touch it is their surfaces that contact. While this may be thought of as a single interaction, it is more correct (and productive) to recognize that there are *two* things happening between surfaces in contact: They do not move through each other, perpendicular to the surface of contact; and They resist moving against each other, parallel to the surface of contact. For this reason we *model* this single interaction with two separate forces: the ***normal***, which models the part which stops the objects from passing through each other; and ***friction***, which models the part which resists the objects sliding across each other.

The Normal: $\vec{n}$

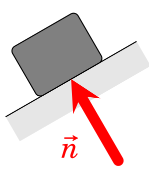
The portion of the interaction that opposes the object moving through the surface is called the *normal*, given the symbol $\vec{n}$. The cause of this force is the physical fact that two pieces of matter can not occupy the same space.

The direction of this force points away from the surface that the object is touching, perpendicular to the surface: in the mathematical terminology, it is *normal* to the surface. (The angle between the normal force vector $\vec{n}$ and the surface is 90°.) If the surface is not horizontal, the normal can not be vertically upwards and can not be sufficient to opposite gravity.

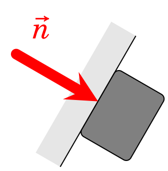
The one thing that you must remember:

the normal is not "mg".

The normal is *not* the opposite of gravity. The normal is the part of the force of contact that stops the object from going *through* the surface. I stress this point because, in my experience, the false idea of "$n = mg$" is usually all that students recall of the normal force from their high school science courses.

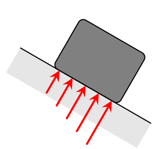


When the surface of contact is not horizontal the normal can not be the opposite of gravity. When friction is small, or absent, the normal and gravity can not sum to zero. This is why things may slide down an incline.

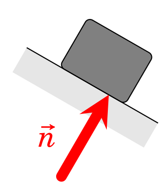


Even when the surface of contact is upside-down and gravity points away from the surface, if the sum of all other forces is pressing the object into the surface, then the normal will still point away from the surface.

For an object touching the surface the normal is a *response* that depends upon all other interactions experienced by the object. If the object is interacting with only the Earth and the surface, then the normal will oppose the *component* of the gravitational force that is trying to move the object through the surface. But if there is also a push or pull, then the normal will oppose the *sum* of all the components of both gravity and the other forces that are trying to move the object through the surface. In all cases the magnitude of the normal acting on the object must be treated as an unknown that must be solved for.



Each piece of the surface that contacts the object exerts a force on the object. These small contributions are distributed across the the area of contact.



The distribution of forces across the surface of contact can be *modeled* by a single force acting at one location. Later, when we study torque, we will see how to determine this location.

In this chapter, for the purposes of finding the sum of forces acting on an object, we can treat the normal as if it were a single force acting at a single location on the surface of contact. In reality the forces of contact are distributed across the surface. A distribution of force across an area is measured as *pressure*, which we will discuss in subsection 1.2.2. A distribution of forces can be modeled by a single force acting at a single location. This single effective force will equal the sum of all the little contributions that are distributed across the surface. In chapter 2, when we study *torque*, we will be able to find and use the location of this single effective force. In this chapter it will be sufficient for us to model the normal as a single force, a push of unknown magnitude, that points perpendicular to the surface.

Friction: $\vec{f}$

Friction is the portion of the surface-surface interaction that is parallel to the surface, given the symbol $\vec{f}$. (The angle between the friction force vector $\vec{f}$ and the surface is 0°.) As the normal opposes the motion of one surface *through* another, friction opposes the motion of one surface *across* another.

Friction is due to two main factors: the roughness of the two surfaces; and the fact that two materials, when in contact, can react chemically to form bonds.

The magnitude of friction is variable. For example, if an object is at rest on a level surface with no other forces acting on it, then the friction required to prevent it from beginning to move is zero. Thus the magnitude of the frictional force is zero. But it you push very lightly on the same object, then friction will grow in magnitude to cancel the applied force and keep the object at rest. (Think how a table does not begin sliding across a room just because you touch it.)

It is productive to think of friction as a question. The force of friction can be found by asking "imagine if there was no friction, which way would the object move across the surface?" The answer tells then tells us that, when there *is* friction, it will oppose that *imagined* motion by pointing the direction opposite. Friction points in the direction parallel to the surface that is required to oppose the motion that would happen if there were no friction. This is important enough that we will repeat it:

Important : Friction is a Question

The force of friction can be found by asking "imagine if there was no friction, which way would the object move across the surface?" The answer tells then tells us that, when there *is* friction, it will oppose that *imagined* motion by pointing the direction opposite.

In this chapter we will exclusively be considering systems and objects that are not moving and that do not start moving. The friction present between surfaces that are not moving relative to each other is called ***static friction***. If the surfaces begin moving across each other, or are already moving, the type of friction is called *kinetic friction*. While these two types of friction are related to each other, they are actually physically distinct. In chapter 4 we will study kinetic friction briefly. In this chapter and in chapter 2 we study static friction exclusively. (The *transition* from static friction to kinetic friction is complicated, and we will not study it in detail in this course.)

In some circumstances friction will be so small that we can completely neglect it. For example, in the human body the friction between the cartilaginous surfaces in a healthy joint is practically zero. It is not exactly zero, but it is so very small in comparison to the other forces acting in and on the body that not including it in our analysis does not effect our results. So read the context carefully: unless you are explicitly informed that the friction can be neglected, you should assume that there is friction present.

The relation between the normal and friction

It will be important to remember that even if the friction is zero between two surfaces in contact, the normal must still be there. There might be little resistance to the surfaces moving across each other, but that does not permit them to move *through* each other. The normal will still oppose that.

However the converse is not true. If the normal is zero, then the friction *must be zero*! The only way that the normal between two objects can be zero is if the object are not touching. If they are not touching, there can be no friction.

In general friction is controlled by the roughness of the two surfaces in contact and their chemical composition. But, experimentally, it is found that there is one other variable the controls the "strength" of the frictional force: the magnitude of the normal force.

Experimentally it is found that if two surfaces are pressed together the friction between them is increased by increasing the magnitude of the normal.

1.2.2 Pressure

In diagrams forces are usually drawn as arrows: lines that touch the object at a single point. In reality, with contact forces this is never the case. When two objects touch there is a shared *area of contact* on their surfaces. The force of contact is distributed across this area. *Pressure* is defined as the measure of how a force is distributed across a surface:

$$\text{pressure} = \frac{\text{force}}{\text{area}} \tag{1.1}$$

The units of pressure are the pascal: $1\,\text{Pa} = 1\,\text{N/m}^2$. In chapter 3 we will study effects of pressure on solid materials. (We will also see how the idea of pressure can be generalized to cases where the force acting is not just pushing.)

In the simplest case of a uniform pressure exerted on a flat surface, the magnitude of the net force exerted by the pressure is

$$\text{force} = \text{pressure} \times \text{area} \tag{1.2}$$

Gasses and Liquids

Pressure is defined as the measure of how a force is distributed across a surface. For solid objects this measure is at the shared area of contact. But in the case of fluids and gasses (like air and water) the pressure that happens at the boundary of the substance (like the inner surface of a container, or the walls of a pool) is due to the motion of the molecules which are bouncing off the boundary. It is the accumulation of these molecular rebounds that exerts a pressure. The pressure at the boundary is thus due to the motion of the molecules *in the volume*.

▲FIX: To be written: At the bottom of a pool, or the bottom of the atmosphere; supporting the weight of the fluid above. Directionless! Remove a half-space, and get the force exerted perpendicular to that surface.

The role of Pressure in sound waves in chapter 5.

Buoyancy

Carving out a piece of the fluid. The pressure at the surface of the carved-out volume. How the sum of all the forces acting at that surface support the volume of fluid. That is the buoyant force.

Replace that volume of fluid with an object. We say that the object has *displaced* the fluid. If there is a difference between the weight of the object and the weight of the fluid that was displaced, then the buoyant force acting at the surface of the object and the force of gravity acting on the object do not sum to zero.

▲FIX: Diagrams

1.2.3 Tension: $\vec{T}$

You know that you can't push with a string, only pull. If you try pushing a string it simply collapses and crumples, and the force you exert at your end is not transmitted to the other end. In contrast if you pull on a string

Tension in a rope (or string, or cable, or chain, etcetera). The force that each piece must exert on its neighbor to remain connected together.

The "tension in the rope" versus the force that the rope exerts on the object attached at its end.

You can't push with a rope (it buckles). The way in which tension can be redirected by pulleys (below, and later in the examples).

Springs and Elastic Objects

The way in which forces can deform solid objects. Springs and deformable surfaces (example: mattress). The force exerted on the object to cause its deformation versus the (reaction) force that it exerts back. Careful!

Muscles

Here's an interesting fact: muscles can't push. Levers and torque (the subject of chapter 2).

Pulleys

▲FIX: HERE: Changing the direction, but not the magnitude, of the tension in a rope, string, cable, or chain.

The *idea* of a pulley is a machine that changes the direction of tension in a rope without changing its magnitude. Usually a pulley is a circular piece of material with a groove around its circumference. The pulley is kept in position, but allowed to turn freely, by an axle through its center. The string or rope (or chain) is kept on the circumference of the pulley by the edges of the groove.

■**PICTURE:** Pulleys:

[[diagram: pulley with rope wrapped over it]]

[[diagram: cross-section of previous]]

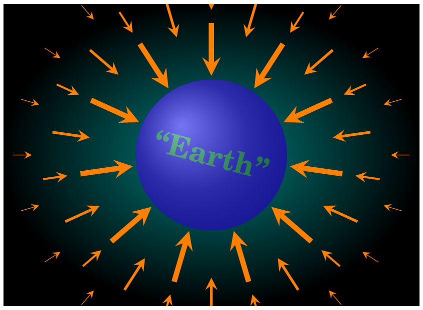
In the human body there are a few portions of anatomy that function as pulleys do, redirecting forces. The patella (the kneecap) redirects the forces of the quadriceps (thigh muscles) over the knee to pull on the tibia (the major bone of the lower leg). The tendons through the wrist redirect the forces of the muscle groups in the forearm to the fingers. In none of these are there a circular disk, but the description of a redirected force is common with the mechanical pulley.

In the examples in sub-section 1.5.3 we will examine systems of more than one pulley, and sometimes with multiple ropes. In such cases we have to consider each pulley itself as an object in order to determine the tensions in the rope, or ropes.

1.2.4 Gravity: $\vec{F}_{\mathrm{G}}$

Every atom in the universe is attracted to every other atom in the universe by ***gravity***. The property of matter that causes this attraction is mass. These forces are weaker between atoms that are further away. The force that the planet Jupiter exerts on you is not exactly zero, but that planet is so very far away, that force is so much smaller than any force that you could feel or measure that you can think of it as being zero. There is even a gravitational attraction between you and your cellphone! But even you breathing on it exerts a repulsive force billions of billions of times stronger than that gravitational attraction. In your life the only gravitational interaction that matters will be with the planet Earth.

The Earth attracts without touching. As I said earlier in this chapter, when you are not touching the Earth is when you become most aware of gravity! This is because the space surrounding the Earth is changed by the presence of the Earth. The words “the space surrounding the Earth” do not just mean “outer space”, beyond the atmosphere. “The space surrounding the Earth” means all positions outside the material of the planet including positions in the atmosphere, near the surface. This property of space itself is called *the gravitational field*.



The gravitational field surrounding the Earth.

This property has, at each point in space, a magnitude and direction: it is a vector. The symbol for the gravitational field is $\vec{g}$. Near the surface of the Earth the magnitude of the gravitational field vectors is

$$g = 9.81\,\mathrm{N/kg} \tag{1.3}$$

Three important things to note about this quantity:

- This is the magnitude of gravity's strength, and is a positive number;
- In your calculations treat this as an exact number, not a number with only three significant figures[1]; and
- You may or may not have seen the units of g written this way before (N/kg), but know that it is equivalent to $\mathrm{m/s^2}$.

Gravity is an attractive interaction between masses. The Law that determines the gravitational force acting on an object is

$$\vec{F}_{\mathrm{G}} = m\,\vec{g} \tag{1.4}$$

where $\vec{g}$ is the gravitational field at the object's location. The magnitude (mg) of this force is called the ***weight*** of the object. The mass of an object (measured in kilograms of mass) and the weight of an object (measured in newtons of force) are proportional to each other, but are not equal since they measure separate physical properties.

the Center of Mass

Similar to pressure, we can model the affect of gravity (which is distributed across every piece of the object) as a single force acting at one point in the object. The location of this point is called the *center of mass* of the object. How to draw gravity acting on an object.

As long as the shape of the object remains constant over time, the position of this force in the object will also remain constant. For uniformly dense objects with simple shapes (like spheres, cylinders, and rectangular solids), the center of mass is at the obvious place: the geometric center of the shape. Determining the location of an object's center of mass is something we will be able to do after we have studied *balance* in chapter 2.

Mass versus Weight

The weight of an object is defined to be the amount of *force* required to hold it up against gravity. When gravity is the only force being opposed, weight and mass are proportional to each other, and that proportionality is a constant: $w = mg$. For this reason in day-to-day, non-scientific contexts a person's mass is usually referred to as their "weight", and "weighing" someone is to measure their mass.

This is technically incorrect, but everyone understands it by the context. There are, however, some contexts where this correspondence between weight and mass fails.

The one common-place context where weight is *reduced* is when an object is placed in water. When immersed in water a *portion* of the force of gravity is countered by the buoyant forces exerted by the pressure of the surrounding fluid. If we call the mass of the object m_{o} and the mass of the displaced fluid m_{f}, then the object's weight is

$$w = (m_{\mathrm{o}} - m_{\mathrm{f}})\,g \tag{1.5}$$

[1]The exact value adopted by the Bureau International des Poids et Mesures (BIPM) is 980.665 $\mathrm{cm/s^2}$ (see `https://www1.bipm.org/fr/CGPM/db/3/2/`). We will use this value rounded to three digits.

This is the vertical component of the force required to keep the object sinking (or rising) in the fluid.

In this case, when the mass of the object is greater than the mass of the displaced water, the force of gravity and the net force due to the pressure acting at the surface of the object sum to a force pointing downwards, and the object sinks. If the mass of the object is *less* than that of the displaced water, then the "weight" (as defined above) becomes *negative*. This indicates that the net force due to the pressure (which points upwards) is larger than gravity acting (downwards) on the object, and the sum of forces acting on the object is upwards, causing the object to rise.

When the object is completely below the surface of the water, the volume of the displaced water is equal to the volume of the object. Expressed in terms of the volume V of the object, and the densities (kg/m^3) of the object ρ_o and fluid ρ_f, the weight is

$$w = (\rho_o V - \rho_f V)\, g \tag{1.6}$$

An astronaut on the Moon has the same mass they had as on Earth, but their weight is less since gravity is not as strong on the Moon. The same would be true of an astronaut on Mars.[2]

Free-fall. Orbit. Physiological changes.

1.3 Equilibrium

Isaac Newton's name is remembered because of his enormous contributions to a variety of sciences: optics, astronomy, calculus, gravity, to name a few. But his most impactful contribution to the physical sciences were his Laws of Motion. In our Age these Laws underlie the engineering of every object and device, from skateboards to airplanes, from metro trains to bottle-tops, and from submarines to satellites. These Laws apply in all systems and to matter in all its forms. In the context of this course we will be studying how these Laws apply to the human body.

I will be presenting Newton's Laws in the reverse order. I am doing this deliberately because it will help emphasize the aspects of each Law that are of importance to us in this course.

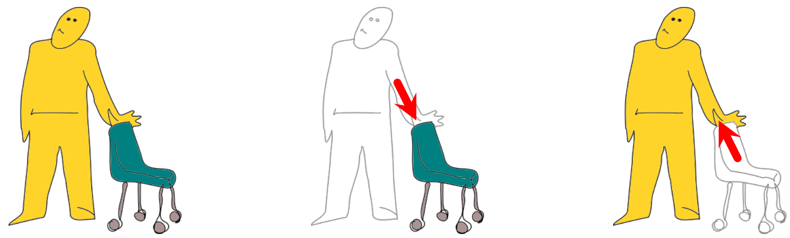
1.3.1 Newton's 3rd Law

The common-knowledge version of Newton's 3rd Law is "Every action has an equal but opposite reaction." This Law embodies the idea of *interaction*.

A force is exerted by some thing onto some other thing; a force is the affect of one on another. A force can not be acting if there is no cause, and a force has no meaning if it is not acting on something. A force is how an interaction between two things affects *one* of the things. When speaking of forces it is critical to be clear about *which thing is the object*.

[2]It may seem nonsensical to want to discuss gravity on the Moon or on Mars. But note that several nations have are currently working towards returning to the Moon and establishing bases there. This will very probably happen in your lifetime.

The goal of this course is to prepare us to be able to think about the human body *physically*. When we are developing our description of a system in terms of the forces acting Newton's 3rd Law will often be a crucial tool. Focusing upon the nature of the *interaction* will, quite often, help us see what the force is and in which direction it is acting. An example that will help us see this is a person pushing a chair.



Object is Chair   Object is Person

Get a chair on wheels and push it away from you. If you are doing this as you would naturally, the chair moves, and you do not. Repeat this, but push the chair using your littlest finger. As you push the chair focus on the fact that your little finger is bending. *You are now aware of the fact that the chair is exerting a force on you*. The force that the chair is exerting on you is equal in magnitude to the force that you are exerting on it. The force that the chair is exerting on you is along the direction opposite to the force that you are exerting on it. The mathematical statement of this, which you may have seen before, is

$$\vec{F}_{\text{A on B}} = -\vec{F}_{\text{B on A}} \tag{1.7}$$

Experiment : Action & Reaction

Stop reading. Pause here and do the little experiment above. Do not read further until you have understood this idea. If it does not make sense, then take notes on any questions that thinking about this raises for you. Bring these questions to class!

The "equal and opposite force" does not mean that the motions of the two interacting objects are also equal and opposite. This is because the motion of an object is determined not by this one force, but by the *sum* of all the forces acting on that object. You exert a force on the chair and the rolling resistance is very small, so the chair begins to roll away. The chair exerts a force on you but the friction between you and the floor is large, so you do not move. If you ask "what is the force?" the answer will depend upon which thing is the object. This is why we must "identify the object".

1.3.2 Newton's 2nd Law

This is the one you'll remember from high school: $\sum \vec{F} = m\vec{a}$. The technical reading of this law is "the acceleration of an object, times its mass, equals the sum of the forces acting on the object". If this were a course in the science program, we would be spending most of the semester focused upon using and understanding this relation between force and mass and motion. You probably saw the application of this law in high school in the contexts of "pushing a box" or "projectile motion". But situations of that type are not of any interest to us in this course.

You may return to thinking about human locomotion in your later courses, like *Kineseology* and the *Management* courses. In this course we will not spend any time with using this specific Law. Besides, in the clinical setting the patient will most likely remain on the treatment bed, and not be flying across the room like a projectile.

Our only interest in the 2nd Law is how it relates to the 1st Law.

1.3.3 Newton's 1st Law

If the sum of forces acting on an object is zero Newton's 2nd Law does *not* say that the object's speed will be zero. It says that the acceleration is zero. It says that neither the object's speed nor direction of motion will change. If it is moving, then it will continue moing, and the way in which it is moving will not be changing. But, if it is not already moving, then it will not start moving. If the sum of forces acting on an object at rest is zero, then it will not start moving.

Newton's 1st Law states that if acceleration is zero, then the sum of forces is zero. It also states that if the sum of forces is zero, then the acceleration is zero.

$$\sum \vec{F} = \vec{0}\,\mathrm{N} \iff \vec{a} = \vec{0}\,\mathrm{m/s^2} \tag{1.8}$$

This Law may seem like a redundancy, since it looks like it is only a sub-case of Newton's 2nd law, but it is a technical necessity: it says that *only forces* are the cause of acceleration.

For us, Newton's 1st Law applied to objects that remain at rest is the only part that is important for this course. If the sum of forces acting on an object is zero, then the object's motion will not change; if it is at rest, it will remain at rest.

1.3.4 An Object in Static Equilibrium

If the sum of forces acting on an object is zero, then it is said to be in *equilibrium*. If it is moving, then its velocity (speed *and* direction) will not be changing. A case of constant velocity is called *dynamic equilibrium*. If it is at *rest* (not moving), then it will remain at rest, and the object is said to be in ***static equilibrium***.

The human body is dynamic. But the ability to maintain static poses (standing, sitting, or holding things) is an important facility. In this text, to gain a basis in the physics behind *Biomechanics*, we will focus on cases of the body in static equilibrium. (The subject of the motion of the human body will be part of chapter 4, in which we study Energy.)

In the cases that we study our goal will be to think physically about what happens inside the body in response to the forces that are either exerted *by* the body or exerted *on* the body. Since our context will be static equilibrium, we will be using Newton's 1st Law to analyze the systems we study.

1.3.5 Determining an Unknown Force

If we know a few, but not all, of the forces acting on an object, can we determine the other forces acting on the object? If we are given that the object remains in static equilibrium, then Newton's 1st Law gives a mathematical relation between all the forces: they must sum to zero. This will be a set of equations: one equation for each component. If we have as many equations as we have unknowns, then we are mathematically guaranteed that as solution will exist. All we will need to do is apply a little algebra to isolate the unknowns, and then use our calculator to determine their values.

The examples at the end of this chapter show this process in detail.

1.4 *The Process*

When you lift something, it is meaningful to ask "what is the tension in my biceps?" The question will have an objective, quantitative answer. The purpose of this chapter to begin developing a method for finding answers to questions like that.

It is fair to say that "the language of Physics is Mathematics". But that is only true if you wish to work with and obtain quantitative values. *Thinking physically* is reasoning conceptually and qualitatively about how things relate to each other through their positions and interactions. It is only after those relationships have been determined that the correct mathematical descriptions can be selected and used. Those steps where you are reasoning conceptually and qualitatively are where you are ***doing physics***.

If you are going to ask questions like "what is the tension in my biceps?", then you must follow this process to get the answer:

The Process

0. Identify the Object!

1. Identify the forces acting on the Object.

2. Draw the Free-Body Diagram.

3. Separately, for each force acting on the Object:
- draw the coordinates
- draw the force (vector)
- determine the components.

4. Use Newton's 1st Law to write the equations to be solved.

5. Solve the equations for the unknown quantities.

Each of these steps will be explained in detail below. In section 1.5 there are examples showing the Process being used. Following that we will be doing a large number of exercises in class.

GOAL :

One of the goals of this course is for you to develop the ability to think physically about the mechanics of the body. The emphasis is on the conceptual and qualitative reasoning. The quantitative steps that follow we do only to provide an objective check of "are we correct?" Remember to focus your efforts on Steps 0, 1 and 2 since that is where we will develop our ability to *think physically*.

What we do here in this chapter we will refine in section 2.4 when we study *torque*, which is the physical cause of rotation. Since the motion of the human body is defined primarily by its articulations (the *joints*) most of the biomechanical examples and exercises will be in that chapter. Keep in mind that the Process we develop here will be same process we apply in those situations.

1.4.0 Identify the Object!

As we saw in 1.3.1 it is critical to clearly identify what is being acted upon. If we are unclear about this choice, it becomes possible to mix-up cause and effect. This is doubly true in the biomechanical context where we are asking about the reaction on or in the person's body because of forces that they are exerting on something else.

Be very clear and specific about what is the object. Only after you have specified the object can you expect a clear answer to questions like "what is the force?"

Write it down: *what is the object?*

1.4.1 Identify the Forces

The purpose of this step is to name what forces are acting on the object. There is no algebra, trigonometry or arithmetic to perform in this step. Here is when you ***think physically*** about the situation.

Think about this: what is interacting with the object? In this step you should first *name* these interactions. The Earth is interacting with the object, obviously, so there is gravity acting on the object. But is the object touching a surface, or surfaces? If yes, then there will be normal (or normals, if more than one surface) acting on the object. Depending upon the details of the situation, there might be friction. Are there any strings, ropes, or chains attached to the object? Then there will be a tension (or tensions) acting on the object. Is there some other object, or a person, touching, pushing, pulling or in some way interacting with the object? Can you name what kind of forces those exert? (It is acceptable to just call such forces "$\vec{F}$" with no special name or symbol).

Write it down: *what are the forces acting on the object?*

Once you've written out this list forces keep it where you can see it. You will check what you do in the later steps against this list.

1.4.2 Draw the Free-Body Diagram

The idea of a *Free-Body Diagram* (abbreviated "FBD") is that the motion of the object is affected only by the forces that are acting on it. Once we have found the magnitude and direction of these forces, the actual *causes* of each of these forces *does not matter*. For this reason we can draw the object *by itself*, the forces acting on it, and then forget about everything else. The resulting diagram is visual representation of the *geometric* information that will be needed to solve for any unknown forces.

To prepare for what we will be doing in chapter 2 draw the FBD as a cartoon of the Object, with the forces drawn where they are acting on the Object. For the question of "do the forces sum to zero?" the location of the forces does not matter. But in chapter 2, the location of the forces is *critical*. Practice drawing the FBD this way.

Using your choice from Step 0 and Newton's 3rd Law you can determine the direction of the forces exerted in an interaction. Be clear to yourself about what is the object to find the direction of the force acting on it due to an interaction. (Think back to the "pushing on a chair" experiment from 1.3.1 as an example of how this kind of thinking finds the correct force.)

Drawing?

Drawing is something I see students struggle with, all the time. As I said at the very beginning of this text, I do not expect you to become physicists. Neither do I expect you to be proficient as *artists*. I will not be judging your diagrams aesthetically. And (here's the hard part:) ***neither should you***.

The purpose of the Free-Body Diagram (FBD) is to bring together in one place all the geometric information about the object's interactions, *qualitatively*. You do not need to use ruler or protractor; a simple sketch is the goal. It does not have to be "beautiful", it just has to show us what we need.

Think Physically!

After you have drawn the FBD, go back and check the list you made in Step 1. Did you include each interaction you listed? This is the place to stop and think physically about the situation. Having drawn the object do you now see if you forgot an interaction in Step 1? This is the place to stop and check for consistency.

Before moving on to the next step you should do one small qualitative check of your results here: Sketch the sum of the vectors you have drawn on your FBD. If the object is in static equilibrium, then these vectors must sum to zero.

If they can not be made to add to zero, then look carefully at the direction of each force, and ask yourself if they might be pointing in a different direction. (In the examples in section 1.5 we will explore how to determine the direction of friction, when it is present.)

If you have confidence in the directions, then play with the magnitudes. Remember that the FBD is a qualitative representation, and that the magnitudes you drew were guesses. Can you make the forces sum to zero by making one of the force vectors longer or shorter without changing their direction? Try it!

Take your time here, as this the part where you *think physically* about the situation. Steps 0, 1 and 2 are where are where you are ***doing physics***. This is so extremely important that I will repeat it here:

GOAL : Thinking Physically

Sketching the forces acting on the object using the Free-Body Diagram is the place where you can ***think physically*** about the situation. Reasoning about the necessary directions and magnitudes of those forces is where you are ***doing physics***. It is one of the main goals of this course to develop this skill.

While the focus of the course is on these steps to develop our physical reasoning, the later steps that follow (3 through 5) will be used to obtain quantitative answers. That will give us precise and *objective* values that we can use to judge the correctness of our reasoning and process.

A structured approach to drawing a FBD

The "conventional" drawing a FBD presented in textbooks (and practised in high school) is small dot (representing the Object) surrounded by arrows starting on the dot and pointed away from it (representing the forces acting on the Object). This is usually presented as a single step with very little explanation, detail, or depth.

Here I advocate for a much more structured approach. The steps outlined here create opportunities to reason physically about the situation, and to potentially even solve the problem!

Drawing a Free-Body Diagram (FBD)

2A. Draw a cartoon of the Object, with each of the forces identified in Step 1 drawn on the object where they act.

2B. Draw the "conventional FBD" with the forces each starting on a dot (which represents the object).

2C. Draw the sum of forces, in counter-clockwise order around the forces in the diagram from Step 2B (usually starting with gravity).

2D. Think Physically to correct the sum, if necessary: Adjust magnitudes; Find friction; etc. Iterate between 2C and 2D until the sum is correct.

2E. Explain physically what the correct sum means for the Object's static stability or trajectory.

This point of this approach is that the "drawing of FBD" is not a single step, but is instead a process that allows us, with just a few simple sketches, to reason physically about the problem at hand, and possibly even solve it. The examples in section 1.5 at the end of this chapter will demonstrate this approach in detail.

1.4.3 Components

The goal of this process it determine the unknown forces that are acting on the object. Those forces will be in response to the other forces acting on the object. These forces all relate together because their sum must be zero.

This is the first step where we must do some mathematics, specifically some geometry and trigonometry, to find the components of the forces acting on the object. For each force acting on the object, in terms of the quantities that you do know and the quantities that you *do not know*, write it down: *What are the components of each force?*

There are many examples of this in section 1.5. I will demonstrate examples in class. And, with my guidance, you will practise many exercises to develop the skills and the proper approach to achieving this Step.

1.4.4 Use Newton's 1st Law

Newton's 1st Law of Motion states that an object's velocity is constant if, and only if, the forces acting on the object sum to zero. In the context of this course, where the objects we study are at rest and stay at rest, their velocity is zero and remain zero because the sum of forces on the object is zero.

Newton's 1st Law can be written: $\sum \vec{F} = \vec{0}\,\mathrm{N}$. But, for our practical purposes, it is the components of this equation that are important:

$$\sum F_x = 0\,\mathrm{N} \tag{1.9}$$

$$\sum F_y = 0\,\mathrm{N} \tag{1.10}$$

When we write out the sum of vectors in terms of their components we will then have equations that relate together *numbers*. These numbers we can then determine using algebra and our calculator. This will be how we get quantitative answers.

The work that was done in Step 3 can now be substituted into these equations to produce the mathematics that must be solved. Take your time, and be very careful with the *signs* and the *units* of the components and values as you substitute them.

1.4.5 Solve for the Unknowns

Use algebra to isolate the unknowns in the equations you generated in Step 4. Then use your calculator to determine their numerical values. At the end of this step, when you have solved for the unknowns, go back and have a look at the sketch you made of the sum of forces. Check to see if your result matches with your qualitative predictions.

At the very end of this go back and re-read the statement of the problem. What were you asked to determine? If you were asked to find a mass, then you should check to see if your result has the correct units. If you were asked to find a force, then your result should be a vector, specified either by its components or by its magnitude and direction. Read the statement of the problem to make sure that you have answered what was asked!

1.4.6 Comments on "Solving Problems"

Before we get into the examples, there are a few things about "solving" textbook physics problems that I need to comment on.

“Solving”

Reading textbook examples can leave you with the impression that, if you just follow the steps, you will always go from start to finish and end up with the correct answer. ***This is false.*** Problem solving is an *iterative* process. Part of problem solving is exploring alternatives. In the examples that I have written I have tried to be explicit about this.

Unless you already have some mastery of the subject, or are some type of savant, it is only rarely that you will know the complete answer just by reading the description of a physics problem. More generally you will have incomplete information and you will have to try different possibilities until you can find a consistent description of the situation.

If this does not yet make sense, I can only ask for your patience until we've looked at some examples involving *friction*.

Why practise

Learning is about changing how you think. The most effective way to do that is to start by knowing how you think *now*. So always begin solving a problem by writing out what you think the answer will be like. Yes!, make a guess! But after that, do follow the steps of the Process. At the end, at the end of Step 5, compare your result with your initial guess. Are they different? If yes, how do they differ? Are they different by a few percent? Or are they completely opposite?

This is the place to pause and reflect on how you are thinking about these situations, and figure out what you need to change in your thinking. I'm here to help with that step, but it will go much faster if you contribute towards identifying where you need the help. Doing the exercises is the place where you work on that analysis.

So now, let's get to work.

1.5 Examples

The discussion in the previous section was, admittedly, vague and abstract. We will gain an appreciation for the strength of The Process by seeing detailed examples of its use in specific contexts.

In all of the examples worked on below the Object is some non-specific *thing*. It's maybe just a rock, or a potato, a volleyball, or some huge wad of melted-together Halloween candy. While part of the goal of this course is to prepare you for *Biomechanics* by learning how to thing objectively and mechanically about the human body, human anatomy is far too complex to use as a *starting point* – so we will choose to begin by studying simple geometric inanimate things. It does not matter what the object is in the examples below. What does matter is following how The Process lets us determine the magnitudes and directions of possibly unknown forces that are acting on the object.

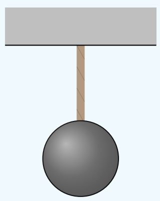
To get the most out of reading these examples I can recommend that you work through them yourself. While reading have a piece of paper next to you and have something to write with. Try each step yourself and check what's done in the example. Remember that problem-solving is an iterative process. Think about, then try something, and then check if it makes sense; go back and try it a different way if some part is not working.

1.5.1 Gravity and Tension

If an object hangs from a single rope, and there nothing else touching it, then the tension must balance the object's weight. In that case the magnitude of the tension must equal mg. If we pretend to be ignorant of the result, we can use The Process to solve for this value. A side benefit of knowing what the answer is before we work the problem is that we can focus our attention on the way that each step contributes towards the answer.

Example 1.1 : Object hanging by one rope

An object of mass $m = 1.37\,\text{kg}$ is hanging from the ceiling by a rope. What force does the tension in the rope exert on the object?



Before doing any work it is always a good idea to ask "what will the answer look like?" In this example we're asked for the tension acting on the object. This tension is a force, so our answer will be a vector with units of newtons. When we get to the end we have to check that our answer has this form.

Step 0: The object is the thing hanging on the end of the rope, represented by the circle in the diagram above. Drawing this picture specifies the geometry of the situation. The geometry is important as it determines the directions of the forces acting on the object.

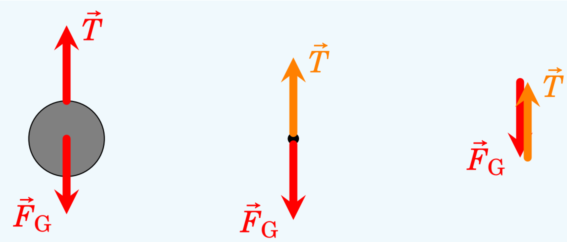
Step 1: The object is interacting with the Earth, and that interaction is the force of gravity. The object is attached to the rope, and the rope is exerting a force of tension on the object. It is important to note that the ceiling is not touching the object. The ceiling is supporting the rope, but it is the rope that is supporting the ball. The ceiling may be the ultimate cause of the object's support, but it is the rope that exerts the force on the object, not the ceiling. There are no other interactions worth including. The object is in static equilibrium, so these forces must sum to zero.

In summary, the two interactions with the Object are

- the Earth, exerting gravity; and
- the rope, exerting tension.

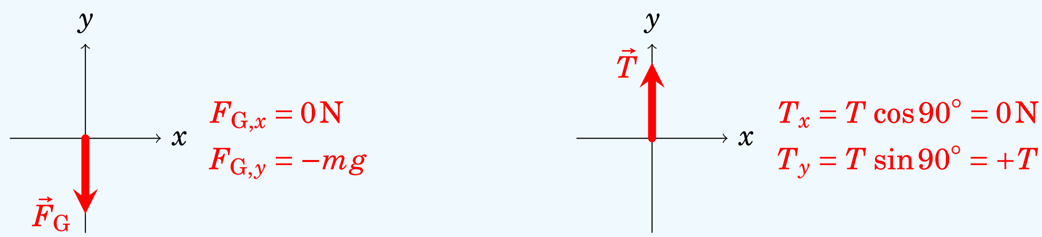
(Always end this step by making an explicit list of the forces.)

Step 2: The Free-Body Diagram is below. Next to that we sketch a qualitative check of the sum of forces.



In this example the check for the sum of forces is simple only because we already have intuited the answer to the question. In later more complex examples we will see how performing this simple-looking step *thoughtfully* provides important information.

Step 3: In this step we choose our coordinates and find the components of each force in the problem. This information will be used in step 4 to write the Newton's 1st Law equations for the system. Our choice of coordinates will be the standard one with the $+x$-axis towards the right and the $+y$-axis upwards.



The components of the tension require some thought and explanation.

Our purpose with this example is to follow how The Process leads us to the correct answer. Our approach, in this situation where we already know the answer, is to pretend that we do not know the answer so we can examine The Process. At this step of finding the components we have to be careful to separate the known from the unknown. We do not know the *magnitude* of the tension, the value of the number T. That is the unknown we are trying to solve for.

What we do know is the *direction* of the tension. Since the object is hanging from the ceiling in static equilibrium we know from experience (and experiment, if you insist!) that the rope will be vertical. The rope can only pull on the object. Thus the tension $\vec{T}$ is vertically upwards, along what we chose to be the $+y$-axis. Using the rules of vector components we find that there is no horizontal component ($T_x = 0\,\mathrm{N}$) and the vertical component is $T_y = +T$. (Read that equality very carefully, and review subsection 0.3 as necessary to understand what quantities are being related.)

Step 4: Since the object is in static equilibrium Newton's 1st law of motion requires that the forces sum to zero. In this situation this becomes the equation

$$\vec{F}_G + \vec{T} = \vec{0}\,\mathrm{N} \tag{1.11}$$

The components of this equation are

$$F_{G,x} + T_x = 0\,\mathrm{N} \tag{1.12}$$
$$F_{G,y} + T_y = 0\,\mathrm{N} \tag{1.13}$$

Now we can look back to the components that we found in Step 3. All x-components are zero, so that equation is just "0 = 0" and does nothing for us. The equation for the y-components is

$$-mg + T = 0\,\text{N} \tag{1.14}$$

Though it is possible to substitute values here, there are two things you should know.

First, the process of substituting values is part of Step 5; it is part of solving for the numerical values of the unknowns. Try, as much as possible, to keep the parts of that Step separate from the parts of this Step.

Second, it is a good habit to keep the equations expressed in terms of symbols instead of numbers before you do the algebra required to solve for the unknowns. Writing one or two symbols is much easier than copying around four or five digits because it drastically reduces the possibility of mis-copying digits, and getting the wrong number at the end. Keep the symbols until after the algebra has been done, then substitute values.

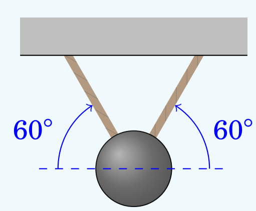
Step 5: If we solve that simple equation we obtain $T = mg$. This says that the magnitude of the tension in the rope (the magnitude of $\vec{T}$) equals the weight of the object (the value of mg), as expected! Using the value given for the mass ($m = 1.37\,\text{kg}$) and the standard gravity ($g = 9.81\,\text{N/kg}$) we find that the magnitude of the tension in the rope is $T = 13.4\,\text{N}$.

Answer: Looking back to the beginning of your work for this example we can check that we were supposed to get a force vector as an answer. So we say that the force exerted on the object by the tension in the rope is 13.4 N, upwards.

The previous example was not challenging to solve, since we already knew the answer! If it is just one rope, then the tension is just the weight. But in the case of multiple ropes each rope will only support a portion of the object's weight. And if the ropes are not parallel to each other they will not just support a portion of the object, they will also have to oppose portions of the forces exerted by the other ropes. In such cases the answers will not be intuitively obvious. In such cases the answers can only be obtained by following The Process.

Example 1.2 : Object hanging by two ropes, symmetrically

An object of mass $m = 1.37$ kg is hanging from the ceiling by two ropes. Each rope makes an angle of 60° with the horizontal. What is the magnitude of the tension in each rope?



In this example we've been asked to find the magnitudes of the tensions in the two ropes. So our final answer will be numbers, each with units of newtons.

Step 0: The object is the thing (represented by the circle) being suspended by the ropes.

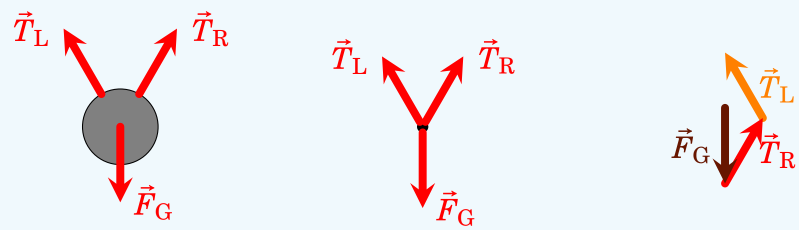
Step 1: The object is interacting with the Earth (gravity), and is attached to the two ropes (tension). Since there are two ropes, there will be two separate tensions. We will call these tensions $\vec{T}_\mathrm{L}$ and $\vec{T}_\mathrm{R}$ for the tension on the left and on the right, respectively. To watch how The Process brings us to the answer we will pretend ignorance and not assume that the tensions are equal to each other. We will also resist any temptation to assume that each tension is "$\frac{1}{2}mg$" (which is wrong!). There are no other interactions worth including. The object is in static equilibrium, so the forces must all sum to zero.

In summary, the three interactions with the Object are:

- the Earth, exerting gravity;
- the rope on the left, exerting tension; and
- the rope on the right, exerting tension.

(Always end this step by making an explicit list of the forces.) It is important to remember that the tensions due to the ropes are separate forces acting in different directions.

Step 2: The Free-Body Diagram (FBD) and the check of the sum of forces is below. (The different colours for the forces are of no significance – they are just so we can distinguish between them visually.)



After drawing the FBD we copied the force vectors over to the side to check if they sum to zero. Oh no! They don't! Did we make a mistake? No. This is not a mistake. This is the place where we uncover something about what is happening *physically* in the system.

We know that the forces must sum to zero. At this point in the problem all we know is the directions of the forces: gravity is downwards, and each tension points along the direction of their respective rope. What we do not know are the magnitudes of these forces. So our check, drawn above, is not a mistake. It is the place where we can figure out the relative magnitudes of the forces in the system.

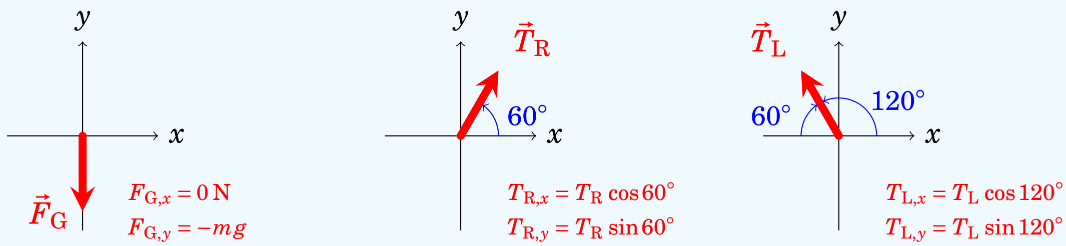
The tension $\vec{T}_\mathrm{R}$ is along the direction of its rope.

The tension $\vec{T}_\mathrm{L}$ is along the direction of its rope.

The tension $\vec{T}_\mathrm{R}$ and the weight $\vec{F}_\mathrm{G}$ must add to something that can be balanced by the tension $\vec{T}_\mathrm{L}$.

Please note that these sketches of the forces do not have to be quantitatively precise. You do not have to use your ruler or protractor to make these sketches. Drawing them free-hand will be enough to find that: (1) each tension is greater in magnitude than half the weight; and (2) the tensions are equal to each other in magnitude. This is important information for us to know so that we can check our results at the end of Step 5.

Step 3: Steps 0, 1, and 2 were us *thinking physically* about the situation. To go further, and get *quantitative* results, we will need to form mathematical equations that describe the situation. Those equations will be the components of Newton's 1st Law of motion. And to write those, we will need the components of each force in the problem:



Note that for the tension on the left we were given the angle measured from the horizontal on the left (from the $-x$-axis). At this step we can use the geometry we've drawn to find the angle measured from the $+x$-axis to use the standard expression for the components of a vector.

Step 4: For this situation Newton's 1st Law has the expression:

$$\vec{F}_G + \vec{T}_R + \vec{T}_L = \vec{0}\,N \tag{1.15}$$

The x-component of this equation is

$$F_{G,x} + T_{R,x} + T_{L,x} = 0\,N \tag{1.16}$$

$$0\,N + T_R \cos 60^\circ + T_L \cos 120^\circ = 0\,N \tag{1.17}$$

The y-component of this equation is

$$F_{G,y} + T_{R,y} + T_{L,y} = 0\,N \tag{1.18}$$

$$-mg + T_R \sin 60^\circ + T_L \sin 120^\circ = 0\,N \tag{1.19}$$

In this system of equations there are two quantities we do not know: the magnitude of the tension on the right T_R and the magnitude of the tension on the left T_L.

Step 5: Solving for the two unknowns in this situation is a case of solving two equations. If you remember algebra from high school we proceed by isolating one unknown from one equation, substituting that expression into the other equation, and then solving the resulting equation for the remaining unknown. It does not matter which unknown you solve for first, the answers will always be the same.

Isolating T_L in the x-component equation gives

$$T_L = -\frac{\cos 60^\circ}{\cos 120^\circ}\,T_R = -\frac{(+0.500)}{(-0.500)}\,T_R = +T_R \tag{1.20}$$

At this step we can see that the tensions, whatever their numerical values turn out to be, will be equal to each other. This is as we might have guessed, given the way the ropes are symmetrically on each side of the object.

Substituting that result into the y-component equation gives

$$-mg + T_{\mathrm{R}} \sin 60^\circ + (+T_{\mathrm{R}}) \sin 120^\circ = 0\,\mathrm{N} \tag{1.21}$$

Solving this for T_{R} gives

$$T_{\mathrm{R}} = \frac{+mg}{\sin 60^\circ + \sin 120^\circ} = \frac{+(1.37\,\mathrm{kg})(9.81\,\mathrm{N/kg})}{(+0.866) + (+0.866)} = +7.76\,\mathrm{N} \tag{1.22}$$

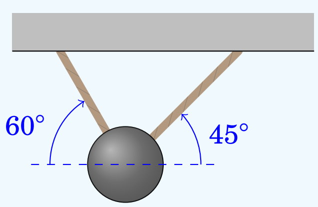
(Here we can pause to notice that this is slightly greater than half the object's weight. It's a little greater than half because each rope is not just supporting half the object's weight, it also has to pull a little horizontally to balance the other rope.)

Answer: The result of Step 5 means that $T_{\mathrm{R}} = T_{\mathrm{L}} = 7.76\,\mathrm{N}$. (We pause to check the statement of the problem, and see that we were asked to find the magnitudes.)

Continuing along the theme of "suspended objects" we can now consider a more general arrangement, where the ropes are asymmetrically supporting the object. Here, obviously, the tensions can not be the same. The interesting question to think about is *which rope will have greater tension*? Is it the rope that is more vertical? Or is it the rope that pulls more to the side? In cases like this, where intuition provides no guide, it is The Process that can lead to the correct conclusion.

Example 1.3 : Object hanging by two ropes, asymmetrically

An object of mass $m = 1.37$ kg is hanging from the ceiling by two ropes. The rope on the right makes an angle of 45° with the horizontal, and the rope on the left makes an angle of 60° with the horizontal. What is the tension in each rope?



Step 0: The object is the thing (represented by the circle) being suspended by the ropes.

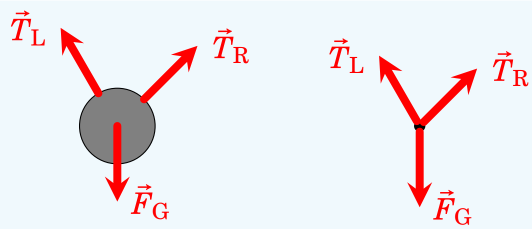
Step 1: As in the previous example, the object is interacting with the Earth and the two ropes. There are no other interactions worth including. The object is in static equilibrium, so the forces must all sum to zero.

In summary, the three interactions with the Object are:

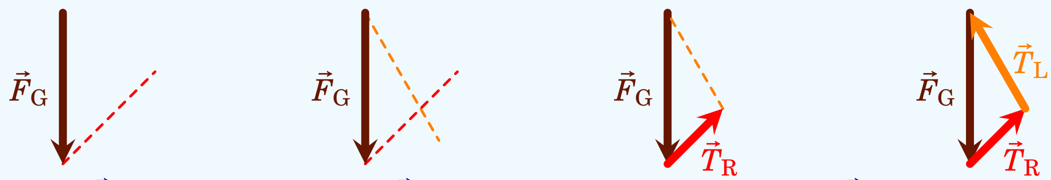
- the Earth, exerting gravity;
- the rope on the left, exerting tension; and
- the rope on the right, exerting tension.

(Always end this step by making an explicit list of the forces.) It is important to remember that the tensions due to the ropes are separate forces acting in different directions.

Step 2: The Free-Body Diagram (FBD) is below.



The check of the forces summing to zero can be constructed in the way shown in the previous example.



The tension $\vec{T}_R$ is along the direction of its rope.

The tension $\vec{T}_L$ is along the direction of its rope.

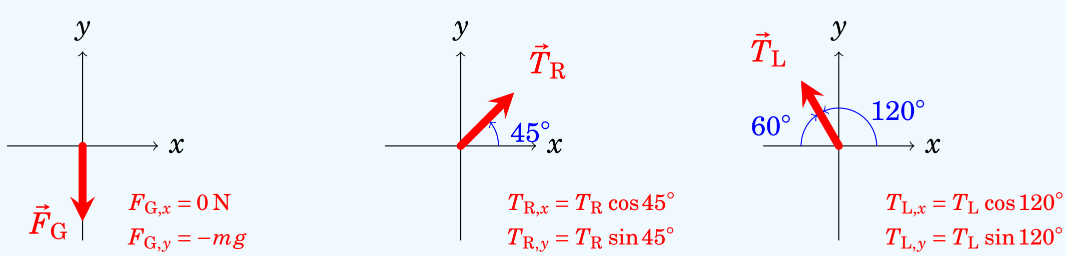
The tension $\vec{T}_R$ and the weight $\vec{F}_G$ must add to something that can be balanced by the tension $\vec{T}_L$.

Here is a good place to pause and take a few moments to *think physically* about the sum of vectors that we just drew. Can we reason about why the tension on the left will be greater?

The only reason there is tension in the ropes is because gravity is pulling downwards on the object. The tensions in the ropes work to oppose that. The rope that is closer to being vertical is better positioned to exert a force along that direction. So it makes sense that the rope that is more vertical (the rope on the left) carries more of the weight.

The rope on the left does not pull vertically only. It is pulling towards the left as well. The rope on the right must balance that horizontal component of force. Because the rope on the right is on an angle, if it has any tension, it will support some of the weight. For these reasons the rope on the right must have some tension, and the rope on the left does not carry all of the weight.

Step 3:



Step 4: Newton's 1st Law for this system is

$$\vec{F}_G + \vec{T}_R + \vec{T}_L = \vec{0}\,\text{N} \tag{1.23}$$

The x-component of this equation is

$$F_{G,x} + T_{R,x} + T_{L,x} = 0\,\text{N} \tag{1.24}$$

$$0\,\text{N} + T_R \cos 45^\circ + T_L \cos 120^\circ = 0\,\text{N} \tag{1.25}$$

The y-component of this equation is

$$F_{G,y} + T_{R,y} + T_{L,y} = 0\,\text{N} \tag{1.26}$$

$$-mg + T_R \sin 45^\circ + T_L \sin 120^\circ = 0\,\text{N} \tag{1.27}$$

In this system of equations there are two quantities we do not know: the magnitude of the tension on the right T_R and the magnitude of the tension on the left T_L.

Step 5: We proceed as in the previous example to solve the system of two equations for the two unknowns. Isolating T_L in the x-component equation gives

$$T_L = -\tfrac{\cos 45^\circ}{\cos 120^\circ}\, T_R = -\tfrac{(+0.707)}{(-0.500)}\, T_R = +1.414\, T_R \tag{1.28}$$

At this step we can see that the tension on the left will be greater than the tension on the right. This was the result of our reasoning in Step 3 when we checked the sum of forces.

Substituting that result into the y-component equation gives

$$-mg + T_R \sin 45^\circ + (+1.414\, T_R) \sin 120^\circ = 0\,\text{N} \tag{1.29}$$

Solving this for T_R gives

$$T_R = \frac{+mg}{\sin 60^\circ + 1.414 \sin 120^\circ} = \frac{+(1.37\,\text{kg})(9.81\,\text{N/kg})}{(+0.707) + (1.414)(+0.866)} = +6.96\,\text{N} \tag{1.30}$$

Using this result in the first equation obtained in this Step, we get

$$T_L = 1.414\, T_R = (1.414)(6.96\,\text{N}) = 9.84\,\text{N} \tag{1.31}$$

As expected from the physical reasoning we did in Step 3 the tension in the rope on the left is greatest. If we had not obtained this result we would be wise to pause, and then carefully check our algebra. Common mistakes to look for are multiplying when we should divide, forgetting a sign, or mis-copying digits in numbers.

Answer: The magnitudes of the tensions in the two ropes (that we were asked for) are $T_R = 6.96\,\text{N}$ and $T_L = 9.84\,\text{N}$.

There are a couple of comments we can make about this result.

If we compare with the results of the previous example (where the ropes were symmetrically on each side of the object) we see that the tension on the left is greater than in the symmetric case, while the tension on right is less than in the symmetric case. We can think of this being a transfer of the weight towards the left.

The most important thing to notice is that the *magnitudes* do not "sum to zero". If we had tried to write something like "$T_R + T_L = mg$", then we would have been completely wrong, and would have no way to solve for anything! (Think back to the 3-4-5 triangle.) This is why we *must* add the forces as vectors (using the components), not as numbers.

1.5.2 Gravity and Contact

Even though objects in contact with surfaces are a more common situation than "object hanging on ropes", we started with examples of gravity and tension because contact forces are more complicated to think about physically and to analyze mathematically.

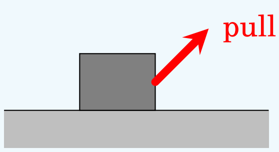
Recall what was said in section 1.2: The normal is not "mg". The normal is the part of the force of contact that stops the object from going *through* the surface. Its magnitude is an unknown that must be solved.

Friction is a question. The force of friction can be found by asking "if there was no friction, which way would the object move?" The answer tells then tells us that, when there *is* friction, it will oppose the *supposed* motion by pointing the direction opposite. Because friction is usually not a given, we may have to *guess* what the forces on the Free-Body Diagram (FBD) are, and then correct ourselves after checking the sum of forces.

The next few examples will illustrate these points.

Example 1.4 : A Heavy Box being pulled, not moving

A box of mass $m = 7.34$ kg does not move while being pulled on. The pull is 72.0 N towards the right at an angle of 45° above the horizontal. Find the magnitude of the normal. Find the friction between the box and the floor that keeps the system in static equilibrium.



We are being asked to find the magnitude of the normal (n) and the force of friction ($\vec{f}$). Our answers will have to be a number and a vector, respectively, each with units of newtons.

Step 0: The object is the box.

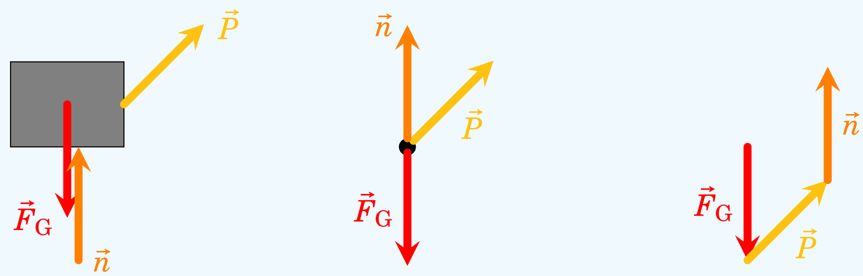
Step 1: The box is interacting with the Earth, the floor, and the rope pulling it. There is gravity, as usual. Because of the rope pulling it there will be tension acting on the box. Because it is touching the floor there will be a normal on the box at surface of contact, and possibly friction. Friction will be non-zero if necessary to keep the box in equilibrium.

In summary, there are three interactions with the object that cause *four* forces:

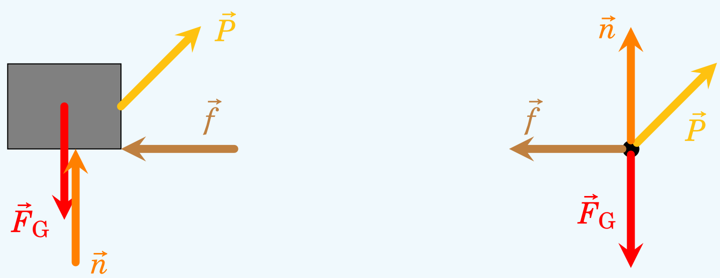
- the Earth, exerting gravity;
- the contact with the surface, which causes
    - the normal (preventing it from moving through the surface), and
    - friction (which acts against it sliding across the surface);
- the pull on the box upwards and to the right.

(Always end this step by making an explicit list of the forces.)

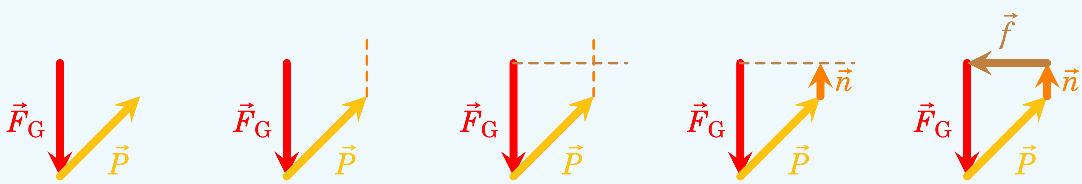
Step 2: When friction is present we must first find which way the object would move *without* friction. Without friction the FBD and check of the sum of forces look like this:



With no friction included we can see that the sum of forces points towards the right. We know that friction will be parallel to the surface. So we can conclude that friction will point horizontally towards the *left* to oppose the sum of all the other forces. We can now correct our FBD:



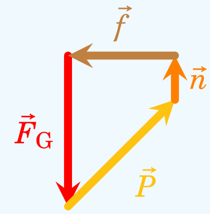
With the correct FBD we can now check the sum of forces. Let's construct it step by step:



The normal is vertical.

Friction is horizontal.

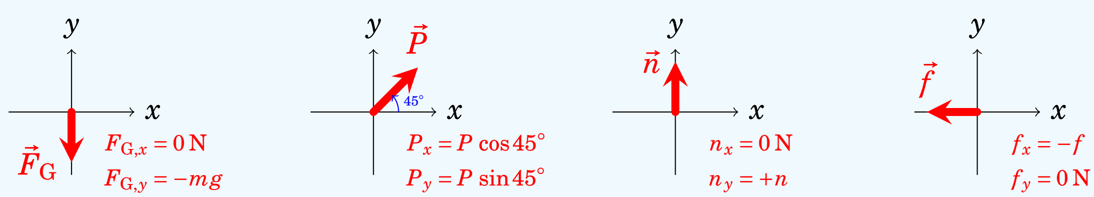
Now that we have the magnitudes correct (qualitatively), let's re-draw this sum of forces a little larger so that we can see how the forces relate. We do this so that we can pause for a moment and think about what is happening *physically*.



Gravity pulls downwards. The tension pulls on the box upwards towards the right. The normal acts to stop the box from moving through the surface. The normal must then counter the sum of the vertical components of all other forces acting on the box. (Since the friction will be parallel to the surface, it does not contribute vertically, and we do not need to know it to find the normal.) The downwards contribution from gravity is partially balanced by the upwards component of the tension. Consequently the normal can be less than "mg" (at the end of Step 5, when we've solved for its value, we can check if this is true). Physically, the tension is pulling the box away from the surface, reducing the pressure on the bottom of the box.

What is friction? *If* there were no friction, the sum of gravity, the normal and the pull would point towards the right, and the object would begin to slide in that direction. Friction opposes that possible motion by pointing towards the left (which we can check once we have solved for it at the end of Step 5). Because the object remains in static equilibrium the sum of all forces must be zero. With the requirement that the friction be parallel to the surface, we can complete the diagram.

Step 3: Before we can write the equations for Newton's 1st law in Step 4 we need to have the components of each force acting on the object.



In this system there are two quantities we do not know: the magnitude of the normal n, and the magnitude of the friction f.

Step 4: Because the box is in static equilibrium, Newton's 1st Law applies. The sum of forces acting on the box must sum to zero:

$$\vec{F}_G + \vec{P} + \vec{n} + \vec{f} = \vec{0}\,\mathrm{N} \tag{1.32}$$

The x-component of the sum of forces equation is

$$F_{G,x} + P_x + n_x + f_x = 0\,\mathrm{N} \tag{1.33}$$
$$0\,\mathrm{N} + P\cos 45^\circ + 0\,\mathrm{N} - f = 0\,\mathrm{N} \tag{1.34}$$

The y-component of the sum of forces equation is

$$F_{G,y} + P_y + n_y + f_y = 0\,\mathrm{N} \tag{1.35}$$
$$-mg + P\sin 45^\circ + n + 0\,\mathrm{N} = 0\,\mathrm{N} \tag{1.36}$$

Step 5: Solving the x-component equation for the friction gives

$$f = +P\cos 45^\circ = (72.0\,\text{N})\cos 45^\circ = 50.9\,\text{N} \tag{1.37}$$

Note carefully that the solved value is the magnitude of the friction. Checking our algebra we find that it is (as it must be) positive. This result means that our choice of direction for the friction was correct. If our choice had been incorrect we would have solved for f and obtained a negative number. That would have warned us that our assumption about its direction was wrong, because magnitudes are non-negative.

Solving the y-component equation for the normal gives

$$n = +mg - P\sin 45^\circ = +(7.34\,\text{kg})(9.81\,\text{N/kg}) - (72.0\,\text{N})\sin 45^\circ \tag{1.38}$$

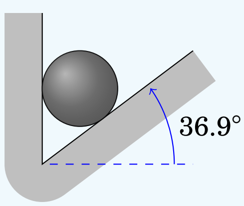
$$= +72.0054\,\text{N} - 50.9117\,\text{N} = +21.1\,\text{N} \tag{1.39}$$

(Remember to not round your results until the last step.) The weight and the tension are essentially the same in magnitude. But since they are different directions they do not cancel each other vertically. The normal makes the difference. Note carefully that, in line with our prediction back in Step 2, the normal is less than the weight.

Answer: Our answers are that the normal has a magnitude of 21.1 N, and the force of friction is 50.9 N towards the left.

Example 1.5 : Sphere wedged in an angled corner

A corner is formed by a vertical surface and a surface inclined by 36.9° above the horizontal. A sphere of mass 4.078 kg is wedged in this corner. At each point of contact between the ball and the surfaces there is a normal force acting. Find the magnitudes of the two normals. (There is no friction between the ball and the surfaces.)



We are asked to find the magnitudes of the two surface-contact normals. We must not expect for either of them to equal the object's weight. We should also not expect them to equal each other!

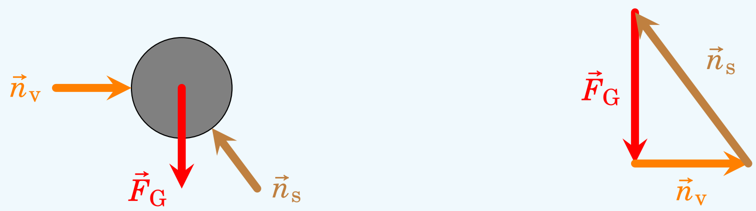
Step 0: The object is the ball that is wedged in the corner.

Step 1: The object is interacting with the Earth (gravity) and is touching two surfaces. At each surface of contact there will be a normal force. We are told explicitly that there is no friction at either surface. In summary, the three interactions with the Object are

- the Earth, exerting gravity;
- the vertical surface to the left, exerting a normal force but no friction; and
- the inclined surface below and to the right, exerting a normal force but no friction.

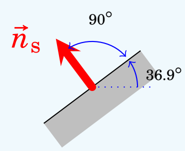
(Always end this step by making an explicit list of the forces.)

Step 2: we will call the normal due to the vertical surface $\vec{n}_{\rm v}$ and the normal due to the sloped surface $\vec{n}_{\rm s}$. The FBD and the check of the sum of forces is as follows:



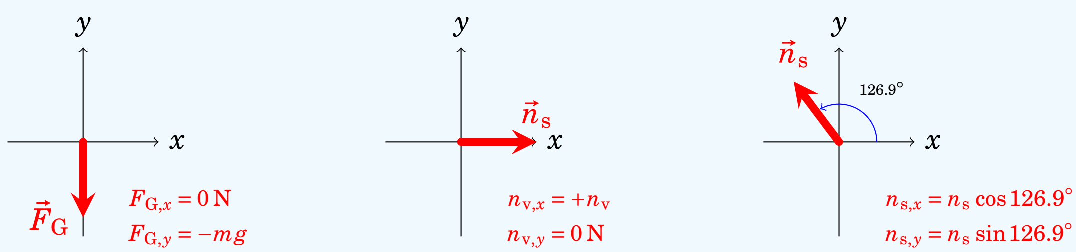
Because the vertical surface exerts a horizontal force the sum-of-forces diagram is a right-triangle.

Step 3: We now find the components of the forces acting on the object. The tricky one to find will be the components of the normal at the sloped surface. Let's have a closer look at the geometry at the point of contact.



The normal is perpendicular to the surface, but the surface is inclined. From the horizontal to the incline is 36.9°, and from the incline to the normal is 90°. Consequently the angle from the horizontal to the normal is $\theta = 36.9° + 90° = 126.9°$.

Now we can write out the components of the known and unknown forces:



Step 4: With the object in static equilibrium Newton's 1st Law requires that

$$\vec{F}_{\rm G} + \vec{n}_{\rm v} + \vec{n}_{\rm s} = \vec{0}\,{\rm N} \tag{1.40}$$

The x-component of this equation is

$$F_{{\rm G},x} + n_{{\rm v},x} + n_{{\rm s},x} = 0\,{\rm N} \tag{1.41}$$
$$0\,{\rm N} + n_{\rm v} + n_{\rm s}\cos 126.9° = 0\,{\rm N} \tag{1.42}$$

And the y-component is

$$F_{{\rm G},y} + n_{{\rm v},y} + n_{{\rm s},y} = 0\,{\rm N} \tag{1.43}$$
$$-mg + 0\,{\rm N} + n_{\rm s}\sin 126.9° = 0\,{\rm N} \tag{1.44}$$

Step 5: The x-component equation has two unknowns in it, while the y-component equation has only one unknown in it. So we solve the y equation first:

$$-mg + n_{\rm s}\sin 126.9° = 0\,{\rm N} \tag{1.45}$$
$$n_{\rm s} = \frac{+mg}{\sin 126.9°} = \frac{(4.078\,{\rm kg})(9.81\,{\rm N/kg})}{0.800} = 50.0\,{\rm N} \tag{1.46}$$

Putting this result into the x-component equation, we solve for n_v:

$$n_\text{v} + n_\text{s} \cos 126.9^\circ = 0\,\text{N} \tag{1.47}$$
$$n_\text{v} = -n_\text{s} \cos 126.9^\circ = -(50.0\,\text{N})(-0.600) = +30.0\,\text{N} \tag{1.48}$$

We note (with relief!) that the magnitude of the normal from the vertical surface is non-negative, as it must be. (Also we note that $mg = 40.0$ N, which means that the sum-of-forces triangle is our old friend the 3-4-5 Pythagorean triangle!) As expected from Step 3 we find that $n_\text{s} > n_\text{v}$.

Answer: The two normals have magnitude $n_\text{v} = 30.0$ N and $n_\text{s} = 50.0$ N. We note that one of these is less than the object's weight, and the other is greater than the object's weight. A fine example of how $n \neq mg$!

Example 1.6 : Person moving a Chair

A person is pushing a chair away from themselves. They are applying a force of 25 N directed 70° below the horizontal. The person has a mass 81 kg. What is the magnitude and direction of the friction acting at the person's feet?

The force we are given in the problem statement is the force that the person exerts on the chair. But we are asked to determine the forces acting on the person. So we must apply Newton's 3rd law to resolve the force that the chair exerts on the person. (Remember the little experiment you were asked to conduct: When you are pushing on a chair, it pushes on you. You should have experienced this by using your little finger to sense the force acutely.)

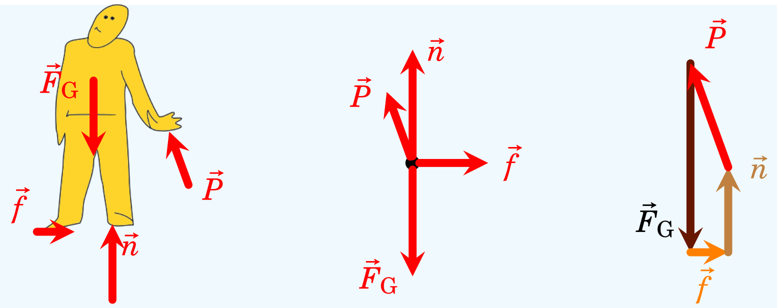
Step 0: The object is the person.

Step 1: The person is interacting with the Earth (so there is gravity), the floor they are standing on (so there is a normal and friction), and the chair (by Newton's 3rd Law this will be the reaction to the person's push). The person is pushing downwards and towards the right on the chair (action). By Newton's 3rd Law the chair is pushing on the person upwards and towards the left (reaction). In summary, there are three interactions with the person (which is the object!) which exert *four* forces on them:

- the Earth, exerting gravity;
- contact with the surface, which causes
  - the normal (preventing them from moving through the surface), and
  - friction (which acts against them sliding across the surface);
- contact with the chair, which exerts a push (the reaction) on them.

(Always end this step by making an explicit list of the forces.)

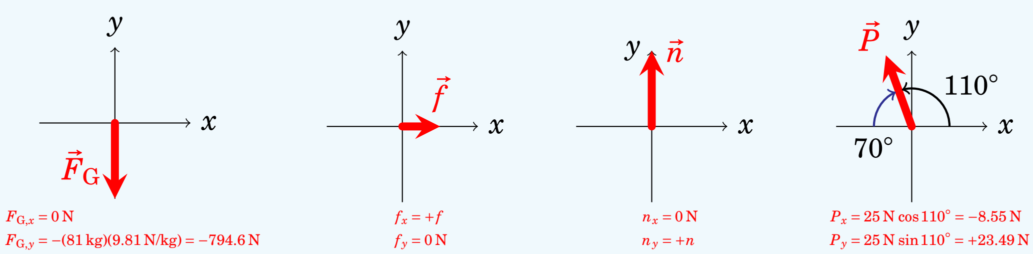
Step 2: The FBD and check of sum of forces:



The check of the sum of the forces shows us that, with the pushing acting on the person towards the left, friction must act on the person towards the right. Most people do not find this result obvious. This process above is what leads us to this conclusion. These steps are where we think about the system, physically. Their importance can not be over-stated.

(As another way to see why the friction on the person must point towards the right, imagine the following: Replace the chair with a parked car, and put the person on a skateboard. If they were to push in exactly the same way as here, they would be rolling towards the left. This means that, if there is friction, it will oppose that motion by acting towards the right.)

Step 3: The components of the known and unknown forces:



The magnitudes of the normal (n) and friction (f) are both unknowns, and our choice of direction for the friction is our educated guess. But we are only asked to find the friction force.

Step 4: This is a situation to which Newton's 1st Law applies:

$$\sum \vec{F} = \vec{0}\,\text{N} \tag{1.49}$$

In this specific situation the sum of forces is

$$\vec{F}_G + \vec{f} + \vec{n} + \vec{P} = \vec{0}\,\text{N} \tag{1.50}$$

The x-component of this equation is:

$$F_{G,x} + f_x + n_x + P_x = 0\,\text{N} \tag{1.51}$$
$$(0\,\text{N}) + (+f) + (0\,\text{N}) + (-8.55\,\text{N}) = 0\,\text{N} \tag{1.52}$$

The y-component of the Newton's 1st Law equation is:

$$F_{G,y} + f_y + n_y + P_y = 0\,\text{N} \tag{1.53}$$
$$(-794.6\,\text{N}) + (0\,\text{N}) + (+n) + (+23.49\,\text{N}) = 0\,\text{N} \tag{1.54}$$

Step 5: we are only asked for the friction. Solving the equation for the x-component we obtain $f = 8.55\,\text{N}$. Note that, if we had assumed the incorrect direction for the friction, we would have obtained a negative number at this step. (Since a magnitude must be positive, that would have signaled that something need to be corrected in some previous step.) Thus we we correct in our physical reasoning in Steps 1 and 2.

Answer: The friction acting at the person's feet is 8.55 N towards the right.

1.5.3 Involving Pulleys

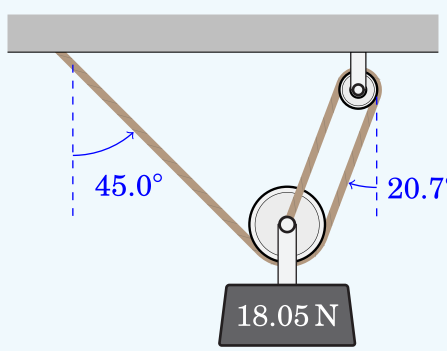
When working on *systems* involving pulleys, there are two issues we must be careful with.

One: Clearly trace out the path that the rope takes through the system, and remember that the tension along that rope has the same magnitude along its entire length. Where a rope makes contact with a pulley is where the rope exerts a force with magnitude equal to the tension in the rope, and along a direction away from the pulley, parallel to the rope. The pulley changes the direction but not the magnitude of the tension in the rope. (In pulley systems when there is more than one rope, carefully trace each one and remember that the tension in each one may be different from the tensions in the other ropes.)

Two: We will always have to carefully think about which pulley or mass will be the object since there will be more than one choice possible. Making this decision may involve doing steps 0 through 2 for each of the possibilities! After that our choice will have to be the object for which there is only one unknown force, or two unknown numbers.

Example 1.7 : A two-pulley system

A block of weight 18.05 N is suspended by the pulley system shown in the diagram. Find the magnitude of tension in the rope. Choose the bottom pulley as the object to solve this problem. (Hint: You will only need to solve the y-component equation of the sum of forces.)



We are asked to find the magnitude of the tension in the rope. Our answer then will be a non-negative number with units of newtons.

That we are told to use the lower pulley as the object reduces the amount of work we have to do, since we would normally have to try *each pulley* as the object to find which one gives the equations we can solve.

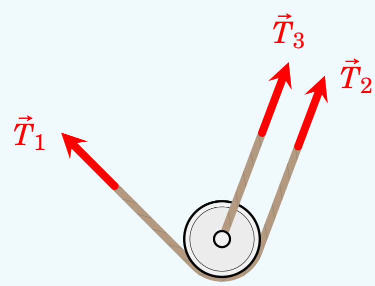
Step 0: In the statement of the problem we are told to use the lower pulley (the one with the block attached to it) as the object.

Step 1: The lower pulley interacts with the Earth, the hanging weight, and the one rope in the system. Let's carefully reason through each contribution.

We list the Earth (as we always must), but, in the case of pulleys, we do not include the force of gravity acting *on the pulley itself*. With pulleys we will always assume that the other forces acting on the pulley (primarily due to the tensions of the ropes touching it) are much greater than the pulley's own weight. Not including the pulley's own weight should not make a significant quantitative difference to our final results.

The 18.05 N block that is attached to the pulley will pull straight downwards on the pulley. (If you don't see why this has to be the case, then consider *as a separate problem* the question of what the forces acting on the block are.)

There is a single rope in the system, but it exerts *three* forces on the lower pulley. The rope wraps around underneath the lower pulley, and so exerts two forces, one at each end of the area of contact between the rope and the lower pulley. The rope then wraps around the upper pulley before returning to attach to the center of the lower pulley, where it exerts the third force.



The diagram above shows this.

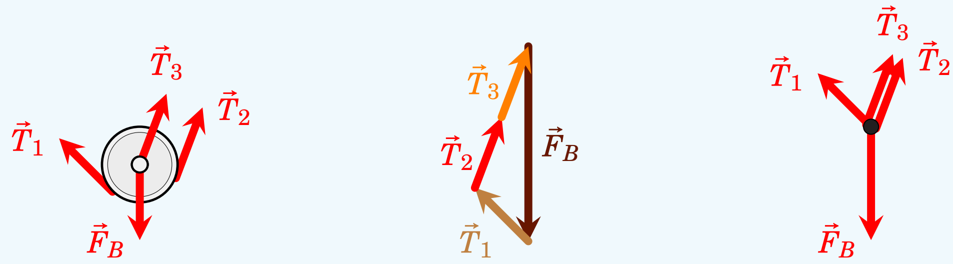
The important thing to remember is that pulleys change the direction, but not the magnitude of the tension in a rope. This means that the three tensions acting on the object all have the same magnitude.

To summarize, the forces acting on the lower pulley are:

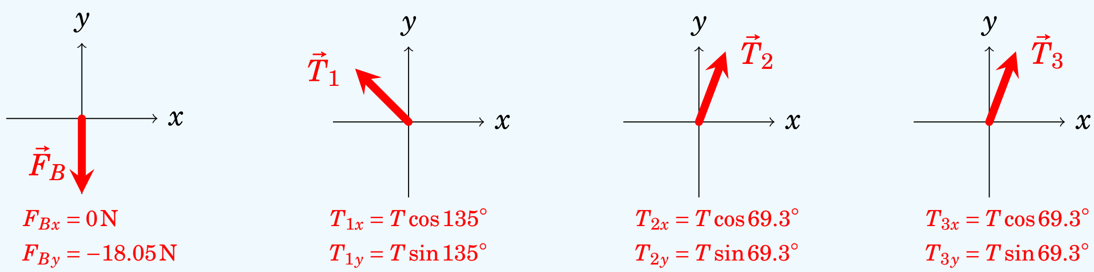
- The tension pulling up towards the left due to the part of the rope that wraps under the pulley (call that $\vec{T}_1$).
- The tension pulling up towards the right due to the part of the rope that wraps under the pulley (call that $\vec{T}_2$).
- The tension pulling up towards the right due to the part of the rope that is attached to the center of the pulley (call that $\vec{T}_3$).
- The attached 18.05 N block pulling downwards (call that $\vec{F}_B$).

In all subsequent parts we must remember that the magnitudes of the three forces due to the rope all have the same, common magnitude: $|\vec{T}_1| = |\vec{T}_2| = |\vec{T}_3| = T$.

Step 2: The Free-Body Diagram for the object, and the check of the sum of forces, is next:



Step 3: The components of the known and unknown forces are next. In what follows we must remember that the three tensions all have the same, common, *unknown* magnitude: $|\vec{T}_1| = |\vec{T}_2| = |\vec{T}_3| = T$. For both $\vec{T}_2$ and $\vec{T}_3$ the angle between them and the vertical is 20.7° (shown in the diagram at the beginning of the problem). If the angle between the y-axis and the vector is 20.7°, then the angle between the vector and the x-axis is $90° - 20.7° = 69.3°$.



Step 4: The sum of forces acting on the object (the lower pulley) is

$$\vec{F}_B + \vec{T}_1 + \vec{T}_2 + \vec{T}_3 = \vec{0}\,\mathrm{N} \tag{1.55}$$

The *hint* was to solve the equation for the y-components:

$$F_{By} + T_{1y} + T_{2y} + T_{3y} = 0\,\mathrm{N} \tag{1.56}$$

$$(-18.05\,\mathrm{N}) + (T\sin 135°) + (T\sin 69.3°) + (T\sin 69.3°) = 0\,\mathrm{N} \tag{1.57}$$

$$(-18.05\,\mathrm{N}) + T(\sin 135° + 2\sin 69.3°) = 0\,\mathrm{N} \tag{1.58}$$

(If, out of curiosity, you were to write-out the equation for the x-components, you would find that $\cos 135° + 2\cos 69.3°$ equaled zero identically, and there would be nothing left to solve in that equation. Give it a try, if you have time.)

Step 5: Solving the y-component equation for the unknown tension:

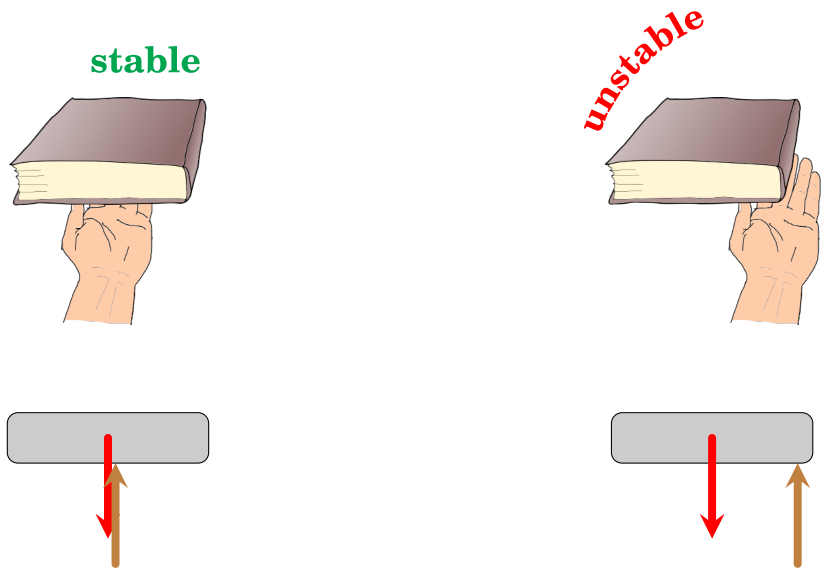
$$(-18.05\,\mathrm{N}) + T(\sin 135° + 2\sin 69.3°) = 0\,\mathrm{N} \tag{1.59}$$

$$T = \frac{+18.05\,\mathrm{N}}{\sin 135° + 2\sin 69.3°} = 7.00\,\mathrm{N} \tag{1.60}$$

Answer: The magnitude of tension in the rope is 7.00 N.

Chapter 2

Torques



If you are trying to support an object you know, from experience, that it will fall over if you support it at its edge. The magnitude of the force you apply doesn't matter. If it is not applied at the correct place, the object will fall over. The sum of forces might be zero, but if the forces are not applied at the correct position on the object, it can not be in static equilibrium. The quantity that relates force and position to equilibrium is called *torque*.

2.1 What is a Torque?

For us, a force is a push or a pull. When we push or pull on something it might move. But, it might also *turn*. When a force is applied to an object in a way that might cause it to turn we can talk about the *torque* being applied to the object. Examples of applying a torque are twisting a door knob to open it, twisting the lid of a jar closed, turning a steering wheel in a car, or the muscles in your forearm turning your hand.

The magnitude of a force measures how rapidly it could change an object's motion. Similarly the magnitude of a torque measures how rapidly it could change an object's *rotation*.

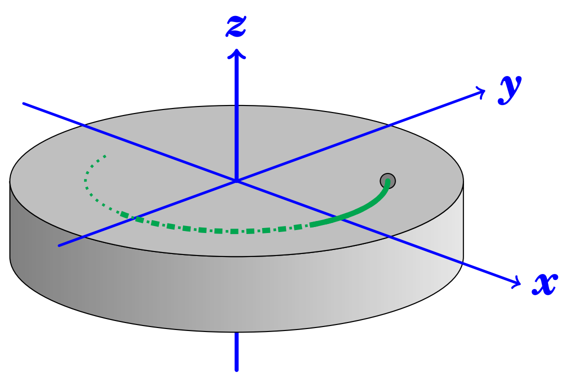
We must remember that even if a force is being applied to an object, that object might remain at rest and not begin moving. An object sitting on the floor is being pulled downwards by gravity, but that force is being balanced by the contact force (the normal) of the floor. Similarly, even if we apply a torque to an object, that object might remain at rest and not begin turning. We might try to twist open a large pickle jar, but that torque is being balanced by the friction between the lid and the jar.

2.1.1 The Axis of Rotation

When an object is rotating, it is moving. This we can agree upon. But that leads to a question that doesn't seem to have an answer: "When an object rotates, in which direction is it moving?"

When an object rotates each atom in the object moves around on a circle. The pieces at the edge move around larger circles than the pieces near the center. The pieces right at the center of the rotation do not move anywhere; their circles are of zero radius.

Each of these circles defines a plane. If we compare any two different pieces in the object their circles are either in the same plane, or are in planes that are parallel to each other. (In the context of this course these planes will be parallel to the xy-plane.) One of the key geometric properties of a plane is that it is defined by a direction that is perpendicular to the plane. Perpendicular to the xy-plane is the z-axis. Each of the circles are centered around this common axis. When the object is rotating in the xy-plane the z-axis is the *axis of rotation*.



In the cases of machinery, devices, tools or other manufactured objects, the axis of rotation is at a pivot or an axle. A door rotates about its hinges (the pivot). A door knob rotates about a shaft (an axle). In cases involving the human body the axis of rotation will usually correspond to a skeletal articulation (a joint). The part of the object that corresponds to the axis of rotation will, generically, be referred to as the *pivot*.

In each of these cases there is some form of connection or constraint that stops the rotating object from becoming separated from the pivot. A bolt stops the door from popping off its hinge; ligaments keep the upper and lower halves of the arm together. This (almost always) means that there will be forces acting on the object *at* the pivot. These forces are referred to as the *forces of constraint* at the pivot.

In the biomechanical context, the forces of constraint at a joint are produced by the forces of contact between the cartilaginous surfaces of the bones on either side of the joint, and by the tension in the ligaments holding the bones against each other. These forces are usually augmented by tension in the surrounding muscles.

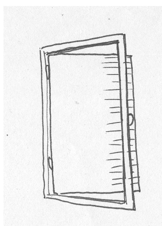
GOAL :

One of the primary goals of this course is to prepare you for thinking about the mechanics of the human body in this way.

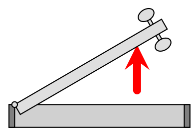
All that being said, we must recognize that the axis of rotation does not always correspond to a mechanical constraint. Think about tipping over chair, or doing somersaults. The object is rotating, but there is no hinge or axle. There is no piece of material holding the object to the axis of rotation. But there still is an axis of rotation. So, just for consistency, even in these cases where there is no mechanical constraint at the axis of rotation, we will still refer to the position of the axis of rotation as the "pivot".

2.1.2 The Strength of a Torque

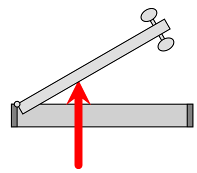
There is an experiment that I need you to do *right now*. Go to a door that you would push to open. Unlatch it so that it will open just with a push. Open it a little by pushing at the edge near the door knob or handle. Then open it a little by pushing at a place near the edge with the hinges. Repeat this using not the palm of your hand, but just one or two fingers so that you are more aware of the amount of force you need to apply.



A door, slightly open...



Opening a door by pushing closer to the handle (further from the hinge) requires less effort.



Opening a door by pushing closer to the hinge (further from the handle) requires more effort.

Do you feel how much more difficult it is to open the door when you push closer to the pivot? The rotation of the door is the same, but the effort you required depended upon the position on the object of your push. If the distance to the pivot was smaller, then the force you had to apply was larger.

Experiment : Torque as force & position

Stop reading. Pause here and do the little experiment above. Do not read further until you have understood this idea. If it does not make sense, then take notes on any questions that thinking about this raises for you. Bring these questions to class!

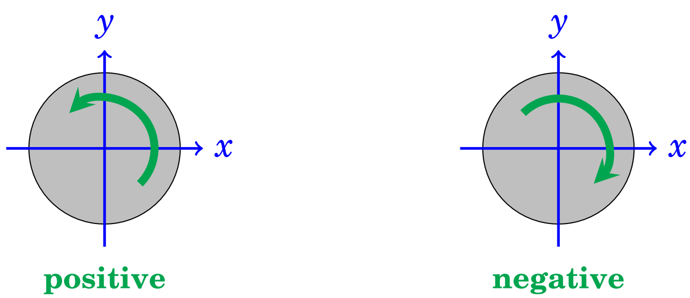
Determining the quantitative value of a torque's magnitude is the topic of section 2.2. For now I want you to try to hold onto this experience as a qualitative guide to your thinking about torque.

2.1.3 The Direction of Torque

If you were studying to become an engineer we would be learning about how torque is actually a vector. This is true because the axis of rotation of an object could be along any direction; north, south, left, right, on an angle, et cetera. The vector of a force points in the direction that the object would accelerate if it were the only force acting on the object. In a similar way, ***the vector of a torque points along the axis of rotation*** that the object would turn about if it were the only torque acting on the object.

In this course we will choose to limit ourselves to cases where the rotation is happening in the xy-plane. This puts the axis of rotation on the z-axis. It also keeps the forces in the xy-plane so that our analysis of forces will follow the methods we developed in chapter 1. This also means that the torque vector will have only one non-zero component: its z-component.

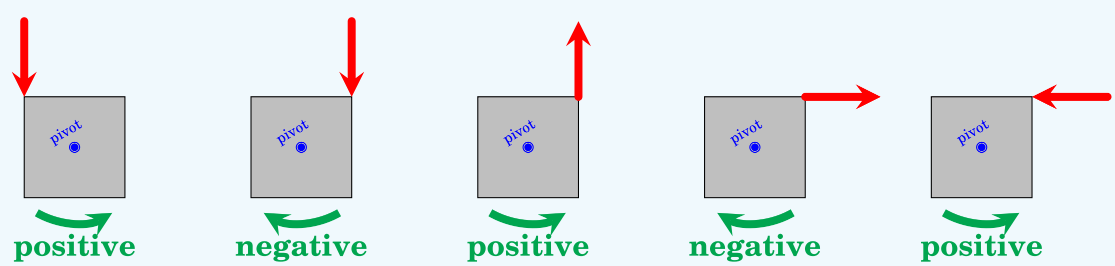
The sign of the torque's z-component is *not* the sign of the component of the force that is producing the torque. The sign of a torque's z-component is the sign of the rotation that it would produce (about the z-axis) if it were the only torque acting on the object. The usual definition for a positive angle in the xy-plane (measuring counter-clockwise from the $+x$-axis) defines the sense of positive rotation.



In the diagram below are some examples for you to think about. Think about the sign of the components of the force that is being applied. And then think about how, geometrically, the resulting rotation relates to the geometry of where the force is being applied. Notice how "the sign" of the force component is, by itself, insufficient to determine the sign of the resulting torque. In section 2.2 we will be developing quantitative methods for determining the sign of the torque from the geometry.

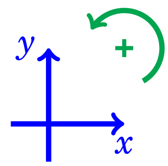
Example 2.1 : Sign of torque in relation to applied force

Below we show some examples of how the sign of the resulting torque relates to the applied force. Note how the rotation that applied force will produce is determined by its relation to the pivot about which the object can turn. (Do you see how each of these results are obtained?)



From now on, when we have to include torque in our analyses, our coordinate system will be x, y and z. The problem is we can't draw in three dimensions, we're stuck in the two dimensions of our page (or screen). So how will we draw our coordinates?

In this course we will choose to limit ourselves to cases where the rotation is happening in the xy-plane. This puts the axis of rotation on the z-axis. With the xy-plane in the page (or screen), the z-axis points outwards at us. Rotation about this axis will either be counter-clockwise (which we consider positive) or clockwise (which we consider negative). The sense of positive rotation is, from now on, part of our coordinate definition. In this specific context we will *represent* the z-axis by noting the orientation of positive rotation *in* the xy-plane. Our axes will look like this:



The arrow pointed towards the right shows the direction of the $+x$-axis. The arrow pointed towards the top of the page shows the direction of the $+y$-axis. The curved arrow surrounding the plus-sign denotes the sense of positive orientation or rotation. A torque that would cause that rotation will have a positive z-component. A torque that would cause rotation in the other orientation will have a negative z-component.

2.1.4 The Symbol for Torque: τ

In the sections that follow we will be finding the quantitative expressions for the magnitude and direction of torque. In those equations and formulas we do not want to have to write the full word "torque" over and over again. We want a simple symbol.

The symbol for torque is the Greek letter "tau":

$$\boldsymbol{\tau} = \text{torque} \tag{2.1}$$

If you want to write the letter τ start with the letter t and just chop the top off. Compare them side by side: $t\tau$.

Torque is a vector $\vec{\tau}$, but our choices of system will, in this course, always have the axis of rotation on the z-axis. So the only non-zero component of the torque will be the z-component: τ_z. Please be very careful to not confuse the magnitude of the torque τ (which is strictly non-negative) with the z-component of the torque τ_z (whose sign *matters*). (Refer back to the sub-section on vector notation in section 0.3 to clarify the difference between these quantities.)

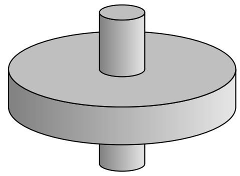
2.2 The Geometry of Torque

In the example at the opening of this chapter the book would fall off our hand if we tried to support it at the edge. The aspect of that example that is important for us to notice is that as gravity pulls the book off our hand it *rotates*. The further our hand is from the center of the book the faster it will rotate. The force acting at the center of the book is gravity. So the factor that controls how fast the book will turn as it falls off our hand is the distance from where it will turn (around our hand) to the place where the force is acting. The purpose of this section is to quantify this simple definition of torque.

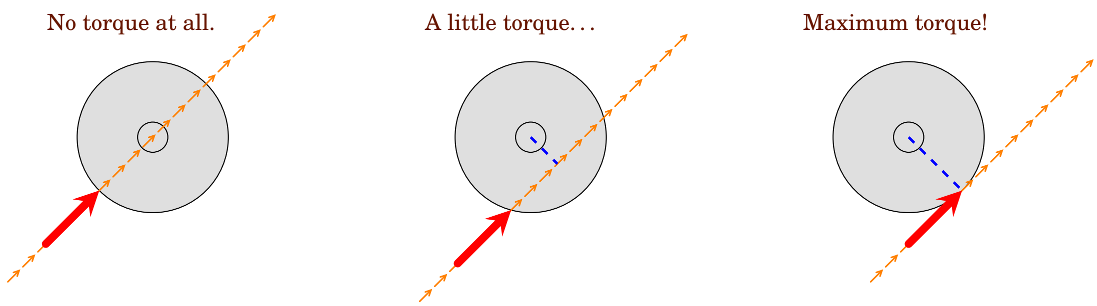
This first step will be to argue qualitatively for the relation between force, position and rotation. In the subsections that follow the relation will be refined, mathematically.

2.2.1 Method: Line of Action & Moment Arm

As a simple mechanical example, let's think about a wheel on an axle, as pictured here:



Looking down onto this wheel from above we can ask which of the three possible ways of pushing on the wheel would make its rotation change fastest:

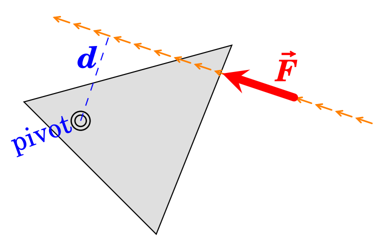


In these diagrams: the red arrow is the force that is being applied to the wheel; the dashed orange line is the line that the force points along, called the ***line of action***; the dashed blue line is the shortest distance from the axis of rotation to the line of action, called the ***moment arm***. Experience (and experiment!) teaches us that the alternative where the moment arm is greatest produces the fastest change in rotation.

For these reasons we ***define*** the magnitude of the torque exerted by a force to be

$$\tau = F\,d \tag{2.2}$$

The quantity F is the magnitude of the applied force. The quantity d is the size of the moment arm (the distance from the axis of rotation to the line of action). This product Fd is the magnitude of the torque.



There are two important facts about the definition of equation 2.2 that we must note and remember when we use it:

Torque : Defined relative to axis

Recall how force has no meaning unless the object it acts upon is specified. Similarly, torque has no meaning unless the axis of rotation (pivot) about which it is acting is specified.

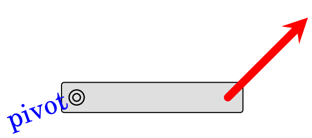
Torque : Units

This definition of torque – a force times a distance – means that torque has units of newtons times metres, written as "N·m". If you remember studying energy and work in high school, then you may think that this should be a joule, written "J". Despite this similarity, the unit of torque is ***not*** the joule. *Why* this is true will be explained when, in this course, we study energy in chapter 4. When writing a value for a torque, write the units as N·m.

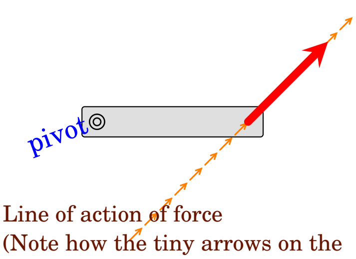
The vector of torque $\vec{\tau}$ is along the axis of rotation. For us, with the line of action in the xy-plane, that is the z-axis. Thus the only non-zero component of torque is the z-component: τ_z. Remember that the *sign* of this component tells us whether the torque would make the object turn clockwise ($\tau_z < 0$) or counter-clockwise ($\tau_z > 0$). So when we write this formula

$$\tau_z = \pm F\,d \tag{2.3}$$

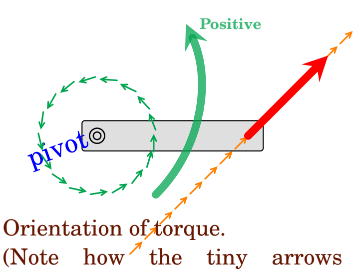
we must determine the sign of the torque (which of the signs "±" we must use).



Force applied to object



Line of action of force
(Note how the tiny arrows on the line of action match the orientation of the direction of the applied force.)



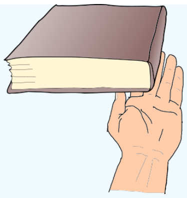
Orientation of torque.
(Note how the tiny arrows around the pivot match the orientation of those on the line of action.)

In sub-section 2.1.3 we saw examples of the relation between the direction of the applied force, the location on the object of the applied force, and the sign of the resulting torque. When the line of action is known, if it is labeled as shown (with little arrows indicating the direction of the applied force), the direction of torque is found by looking at how the line of action passes the pivot.

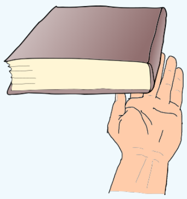
Note carefully that each element that contributes to the determination of torque is relative to the pivot. For this reason we must remember that ***the pivot must be specified*** for a torque to have meaning. Conversely, a torque only has meaning relative to a pivot. The magnitude of a torque measures the strength of a force's ability to change an object's rotation. But that magnitude, and the rotation that might follow, are relative to a pivot (or axis of rotation). Return to the experiment with the door in subsection 2.1.2 to appreciate this.

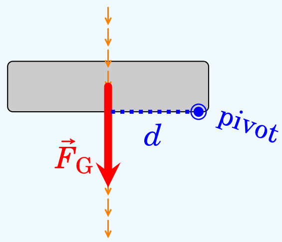
Example 2.2 : Supporting a book

As shown in the diagram below, you are trying to hold up a heavy, hard-cover textbook. It has a mass of 3.00 kg, and is 26.0 cm wide. You are supporting it at its edge. What is the magnitude of the torque exerted by it weight about your hand?



The magnitude of the torque applied by gravity will be the magnitude of the force of gravity, times the moment arm from where the textbook is being supported by your hand to the line of action of gravity. Here's a sketch of the geometry:





When considering torque one of the most important steps is to identify the *pivot*; to explicitly state the location of the axis of rotation about which the torque is acting. In this example the axis about which the textbook will rotate before falling is at the place where your hand is trying to support it; so the "pivot" is at the hand. Gravity acts at the center of the book, so the moment arm is half the width of the book.

The magnitude of the applied torque is

$$\tau = F\,d = (mg)d = (3.00\,\mathrm{kg} \times 9.81\,\mathrm{N/kg}) \times (\tfrac{1}{2} \times 0.260\,\mathrm{m}) = 3.83\,\mathrm{N{\cdot}m} \tag{2.4}$$

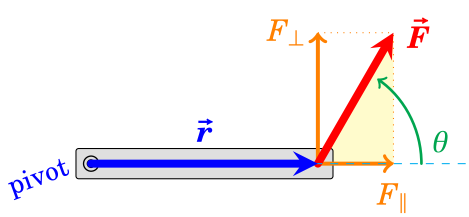
Note carefully that:

- we are given the mass (in kilograms) of the book, but we need to find the *force* (in newtons) exerted on the mass of the book, so must multiply by g
- the units of torque are N·m, so we must convert the width of the book to an expression in metres

Explicitly writing the units of the quantities you are about to use in a calculation will inform you of what (if any) conversions need to be done. Always include your units!

2.2.2 Method: Perpendicular Component

When a force is applied to an object with a pivot we can think of the force as having two components. Components are measured relative to an axis, and in the case of torque the axis the direction defined by the line that connects the pivot to the place on the object where the force is acting.



One component is the part that points either directly at or directly away from the pivot, along the direction that extends from the pivot radially out to where the force is acting on the object. That component does not create a torque since it is either pulling or pushing the object directly away from or towards the pivot. The other component is perpendicular to the radial direction. This component creates a torque since it pushes the object *around* the pivot.

The radial vector $\vec{r}$ points from the pivot to the place on the object where the force is acting. The portion of the force $F_\parallel$ that is parallel to $\vec{r}$ does not contribute a torque since it is either pulling or pushing directly on the pivot. The portion of the force $F_\perp$ that is perpendicular to $\vec{r}$ is the portion that affects the rotation of the object.

In biomechanics the portion $F_\parallel$ of the force parallel to the limb will either be pushing on the limb towards the joint ("compression") or pulling on the limb away from the joint ("distraction"). The perpendicular component $F_\perp$ effects the rotation of the limb about the pivot, but the parallel component $F_\parallel$ *exerts a force on the pivot*. This is a critical aspect to be aware of since forces acting on the joint could aggravate existing damage or cause injury.

2.2.3 Result: Magnitude & Sign of Torque

The torque produced by a force $\vec{F}$ is determined by *where* on the object the force acts relative to the axis of rotation (pivot). The radial vector $\vec{r}$ is from the pivot to the point on the object where the force $\vec{F}$ is acting. When the angle θ between the axis defined by $\vec{r}$ and the line of action of the applied force $\vec{F}$ is known, the component

$$F_\perp = F \sin\theta \tag{2.5}$$

(the part of $\vec{F}$ that is perpendicular to $\vec{r}$) is the portion of the force that contributes a torque. In this F is the magnitude (a non-negative quantity) of the applied force, and θ is the angle measured conventionally with counter-clockwise being positive and clockwise being negative. The component $F_\perp$ is a number whose sign matters.

Since $F_\perp$ acts perpendicular to $\vec{r}$ the magnitude r is the value of the moment arm for the torque that $F_\perp$ exerts. Thus $\tau_z = F_\perp r$, or

$$\tau_z = F\, r \sin\theta \tag{2.6}$$

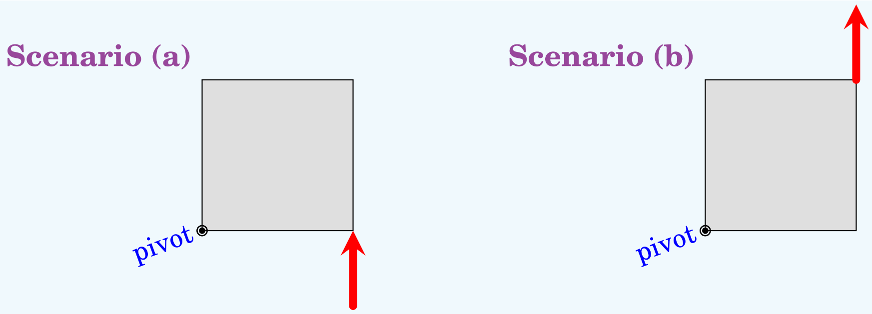
As long as the angle θ is measured conventionally (with counter-clockwise being positive and clockwise being negative) then this equation will always have the correct sign of torque.

Toque of a Push versus Torque of a Pull

Any push or a pull of equal magnitude applied along the same line of action will produce the same torque.

Example 2.3 : Toque of a Push versus Torque of a Pull

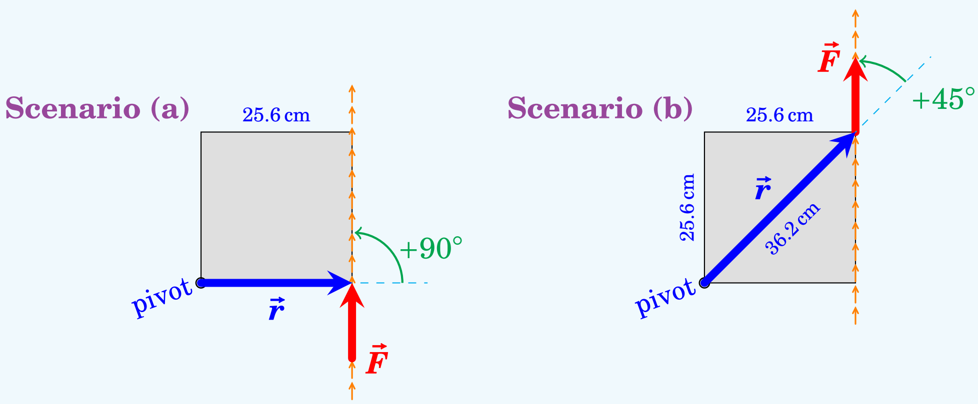
Using the formula $\tau_z = F r \sin\theta$ calculate the torque exerted on each object in the diagram by the force shown about the given pivot. Compare their values, and explain their interrelationship, if any.



The objects are identical cubes of edge-length 25.6 cm. The forces are of equal magnitude (both 74.0 N), and point in same direction (vertically upwards), but are applied at different positions on the objects (as shown in the diagram).

We are instructed to use the formula $\tau_z = Fr\sin\theta$ to calculate the torque. In both cases we know that the magnitude of the applied force is $F = 74.0\,\text{N}$. It thus remains for us, in each case, to find the distance r from the pivot to place where the force is applied, and the angle between that direction and the line of action. We are comparing a vertical push applied at bottom right corner against a vertical pull applied at top right corner.

But, before we calculate anything, we can note that the distance to the line of action is same in both scenarios. With the forces having the same magnitude F the moment arms d being the same, we can expect that the torques will have the same magnitude.



In scenario (a) the distance from the pivot to the bottom right corner is $r = 25.6\,\text{cm} = 0.256\,\text{m}$. The angle between the direction of $\vec{r}$ and the direction of the force along the line of action is $+90°$. This angle is positive because we must turn counter-clockwise to go from the direction of $\vec{r}$ towards the direction of $\vec{F}$. Thus, with r expressed in metres, we obtain

$$\tau_{(a),z} = Fr\sin\theta = (74.0\,\text{N})(0.256\,\text{m})\sin(+90°) = +18.9\,\text{N}\cdot\text{m} \tag{2.7}$$

where the positive signifies a counter-clockwise torque.

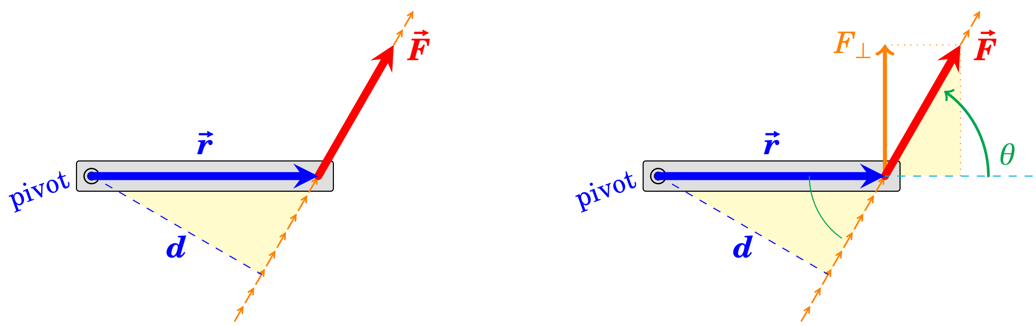
In scenario (b) the distance from the pivot to the top corner is larger by a factor of $\sqrt{2}$ because $r = \sqrt{(25.6\,\text{cm})^2 + (25.6\,\text{cm})^2} = 36.2\,\text{cm} = 0.362\,\text{m}$. But the angle θ is smaller ($45°$ instead of $90°$), so $\sin\theta$ is smaller ($\frac{1}{\sqrt{2}}$ instead of 1), and $r\sin\theta$ has the same value as in the previous scenario:

$$\tau_{(b),z} = Fr\sin\theta = (74.0\,\text{N})(0.362\,\text{m})\sin(+45°) = +18.9\,\text{N}\cdot\text{m} \tag{2.8}$$

Beyond having the same magnitude, as expected, we find that the two torques have the same sign as well. This is to be expected since the line of action is the same in both scenarios and the forces have the same orientation on the line of action.

Equivalence of Methods

The magnitude of torque was defined in terms of the moment arm d. Physically it is the component $F_\perp$ of the applied force that exerts torque about the pivot. These two approaches must give the same predicted value of torque. How does that happen? Consider the diagram below.



In the diagram above the two shaded triangles are *similar* triangles, with the ratios d/r and $F_\perp/F$ both equal to $\sin\theta$ in magnitude. Mathematically

$$\tau_z = F\,r\,\sin\theta \tag{2.9}$$
$$= F\,(r\,\sin\theta) = \pm F\,d \tag{2.10}$$
$$= (F\,\sin\theta)\,r = F_\perp\,r \tag{2.11}$$

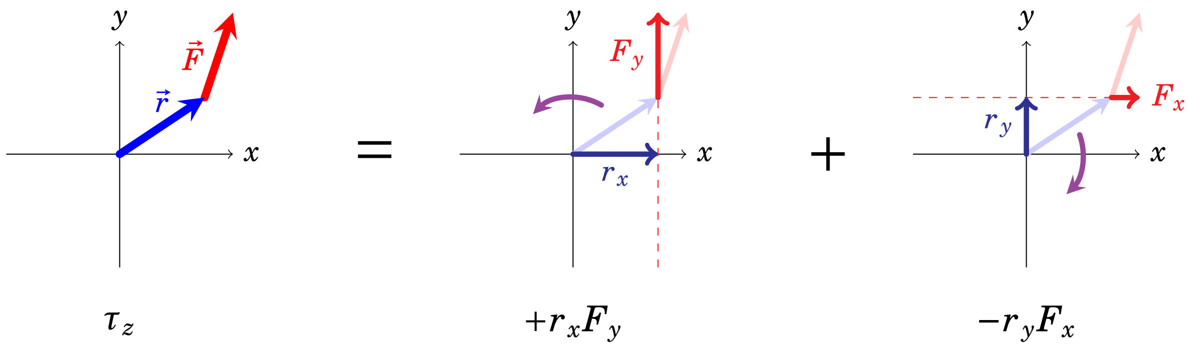
where the choice of $\pm$ is made to match the sign of $\sin\theta$.

These two equivalent expressions have different utility. In cases when we are given or can see directly the value of the moment arm the expression $\tau = \pm Fd$ is the quickest and easiest to use to find the torque.

However, in most biomechanical contexts the value of the moment arm will not be apparent, and will most likely not correspond to a physical distance on the object. Finding the perpendicular component of the force ($F_\perp$) will be both necessary and more meaningful.

2.2.4 Advanced Method: Cartesian Components

If you have the components of the force and of the position vector, there is a way to calculate the force without having to find the angle or the moment arm. This method should be considered as *advanced* and is usually not necessary for use in the context of this course. But it does make explicit the way in which the sign of the torque is determined by the signs of the components of the force producing the torque. You may skip this sub-section on your first reading – unless you are curious...



A force $\vec{F}$ acting on an object at position $\vec{r}$ relative to the pivot exerts a torque. The vertical part of the force (the y-component) acts along a line of action that is a distance r_x from the pivot. The horizontal part of the force (the x-component) acts along a line of action that is a distance r_y from the pivot. The sign of the torque that each of these contributes is shown in the diagram above.

The sum of torques is itself a torque. (This is explored in the next section.) Combining the contributions described in the previous paragraph we find that the component of torque around the z-axis can be written in terms of the components of the applied force and the components of the position vector:

$$\tau_z = +r_x F_y - r_y F_x \tag{2.12}$$

If you already have, or were given, the components of the applied force and its position, then this expression gives you the torque without requiring you to determine any angles or rotational orientations. You can try using this expression by applying it to the examples in sub-section 2.1.3.

2.3 Sum of Torques & Equilibrium

The picture at the opening of this chapter planted the idea: torques contribute to equilibrium. The development outlined here parallels what we did in section 1.3 with forces.

2.3.1 Summing Torques

Torque is a vector, and it follows all the rules of vector summation, as usual. In this course, however, we are limiting ourselves to cases where the axis of rotation is the z-axis. Thus we will only ever have the z-components of the torques (τ_z) to worry about. When we sum torques, we only need sum the z-components. The x and y components of each torque will all actually be zero, with no need to sum them. To get the correct sign and magnitude for each contribution to τ_z we need only follow the methods of section 2.2 ("The Geometry of Torque") to use equation 2.6 ($\tau_z = Fr\sin\theta$) properly.

Torques add as vectors, but there is one critical constraint that must be obeyed for the sum to be meaningful:

CRITICAL : Each torque relative to the same pivot

Torque measures the strength of a force's ability to change an object's rotation. To compare or combine (add) torques each torque must be measured or calculated relative to the same pivot.

This is necessary since torque is the measure of a force's ability to rotate about a specific axis of rotation. That measure is *defined relative to that choice of axis*. Consequently it makes no sense to compare torques due to different force about *different axes*. (This would be similar to comparing forces acting on different objects.)

2.3.2 Equilibrium

Newton's 1st Law for forces states that an object is in static equilibrium when the forces acting on it sum to zero ($\sum \vec{F} = \vec{0}\,\text{N}$). There is a corresponding Newton's 1st Law for torque: If an object is in static equilibrium, then the torques acting on it sum to zero ($\sum \vec{\tau} = \vec{0}\,\text{N·m}$).

In this course (where the axis of rotation is only ever along the z-axis) the equations that, together, are the statement of Newton's 1st Law are

$$\sum F_x = 0\,\text{N} \tag{2.13}$$

$$\sum F_y = 0\,\text{N} \tag{2.14}$$

$$\sum \tau_z = 0\,\text{N·m} \tag{2.15}$$

The sum of forces along the x-axis summing to zero means that the object doesn't move left or right. The sum of forces along the y-axis summing to zero means that the object doesn't move up or down. The sum of torques about the z-axis summing to zero means that the object doesn't rotate (neither clockwise nor counter-clockwise).

2.3.3 The *Choice* of Pivot

If an object is in static equilibrium, then it is not rotating about the pivot. But neither is it rotating about any other axis; the object is not rotating at all. This fact leads to a very powerful statement:

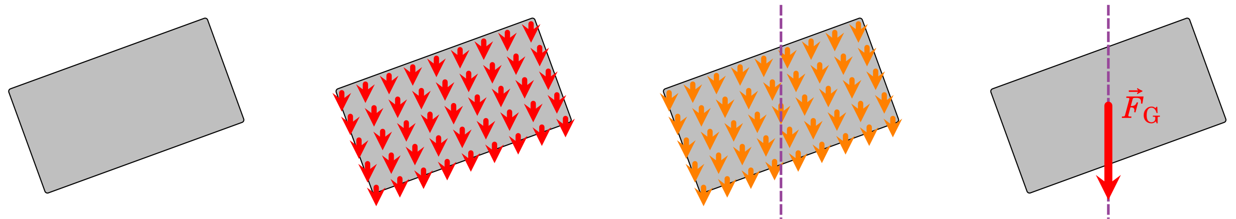
Important : Equilibrium independent of pivot

If an object is in static equilibrium
then the sum of torques is zero
about *any* axis.

Writing out the sum of torques equalling to zero generates an equation that we can use to solve for unknown forces that are acting on the object. The fact above allows us to *choose any axis*, set the sum of torques about that axis to zero, and generate a new equation. When we are writing the equations to be solved the "pivot" does not need to be an actual mechanical pivot! Since the forces at the actual mechanical pivot do not exert a torque about that pivot, this freedom can help think about and solve for those forces. Examples at the end of this chapter will demonstrate how.

2.3.4 Where does Gravity Act?

Every atom in the universe is attracted to every other atom in the universe by ***gravity***. So when we draw a picture of earth's gravity acting on an object it should really be like the drawing second from the left:



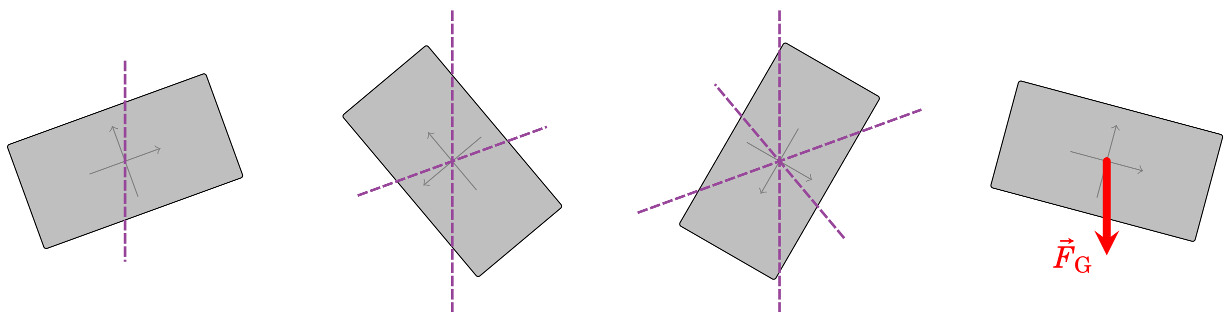
Here is an object near the Earth.

Here are the contributions of gravity acting on each of the pieces of the object. Gravity does not act at *one* point, it acts on *all* points.

If the "pivot"is placed anywhere on this dashed line the torques from forces to the left will be balanced by torques from forces on right, and the sum of torques will be zero about that pivot.

This single force is equal to the sum of all the contributions from gravity acting on all pieces of the object. And like the sum of all the torques from the all of the pieces, this force exerts zero torque about its line of action.

If take the object pictured above and orient it different ways, the lines about which the torques sum to zero will all intersect at a common point. About this point, the ***center of mass*** of object, the torque exerted by gravity is zero, regardless of its orientation.



When analyzing the affect of the force exerted by gravity on the equilibrium (or motion!) of an object it is the *sum* of all the gravitational forces on its pieces that matters; this sum is modelled as a single force that we call $\vec{F}_{\text{G}}$. When analyzing the affect of the *torque* exerted by gravity on the equilibrium (or motion!) of an object it is the *sum* of all the gravitational torques on its pieces that matters; this sum is modelled as a single torque due to a single force ($\vec{F}_{\text{G}}$) that acts at the center of mass.

The Torque exerted by Gravity

If the pivot is placed at the center of mass, the torque on the object due to gravity is zero. If the pivot is placed anywhere else on the object, the torque due to gravity might *not* be zero. The pictures at the opening of the chapter, and example 2.2.1, present this idea.

It is important to remember that, regardless of where the pivot is located, gravity will exert a force on the object. When we are determining the sum of forces on the object, gravity will always contribute. But when determining the sum of torques the relation between the position of the pivot and the center of mass will determine the amount of torque gravity contributes. If the line of action of the force of gravity passes through the pivot, then its contribution to torque will be zero.

2.3.5 Where does the Normal act?

Picture of rectangular object at rest on incline. Picture of the distribution of surface forces, with normal and friction being pictured separately. If the object is stable, then it will not be rotating about *any* axis. Choose the lower corner of the object. All the contributions from friction have lines of action passing through the pivot, and thus do not exert a torque!

We will consider what happens if the surface is gradually tilted. At each step we take the center of mass to be the pivot.

When the surface is horizontal the normal is at the center of the object, pointed directly at the center of mass. The normal and gravity directly oppose each other, sharing a common line of action, and the torques of these two forces would sum to zero about any axis. With the "pivot" at the center of mass the torque of each of these forces is zero.

As the surface is inclined friction becomes non-zero if the object remains in static equilibrium. The contribution to torque from friction has a line of action along the bottom surface, with a moment-arm whose distance does not vary with the degree of incline. *If* the normal were to continue to act at the center of the object its contribution to torque would remain zero. Those contributions (gravity and normal both zero, but friction *non-zero*) can not sum to zero. So we conclude that the normal can not remain at the center of the object if the object is to remain stable. The line of action of the normal must also move away from passing through the center of mass.

When the surface is inclined, if the distribution of normal forces is uniform, then the sum of torques can not be zero. Thus the distribution of normal forces must be *non*-uniform. As we will see in the examples at the end of the chapter, the effective (single force at a point) normal must be located under the center of mass. This means that the distribution of pressure under the object will tend to be greatest nearest the end above which the center of mass is located.

2.3.6 Forces at the Pivot

When a door is wide open and at rest, the hinges must support the weight of the door. If the hinges are too weak (or if too much additional force is applied to the door), then the door can rip out of the wall. When you pick up a very heavy object the shape of the two bones that contact in a joint work to prevent them from separating. But if too great a force is exerted on the joint, the bones can dislocate.

The point to those examples is to make you see that there are forces acting at the pivot to keep the pieces on either side connected to each other even as they rotate around each other. By definition the forces acting at the pivot are zero distance from the pivot, and so can not exert a torque about the pivot. But there must be forces at the pivot to keep the object connected to it.

There is one small detail that, in real-world applications, can make the physics of the pivot contribute: when there is rotational friction in the pivot. If you can picture the surfaces of the joint, bearing, or hinge (whatever the pivot is) imagine the surfaces of contact rotating past each other. If there is friction where these surfaces meet, then there will be a torque exerted by the pivot! Usually (typically!) even when there is friction present in the pivot, it has a very small moment arm (the distance d) and the resulting torque is small in magnitude.

2.4 *The Process*

What we do here is a refinement of the Process introduced in section 1.4. Go back and quickly review that section before going any further!

The purpose of the Process here, when torques are considered, is essentially the same as previously: To provide a systematic method for solving for unknown *forces* acting on objects in equilibrium. The one difference now is that, with torque relating force and position together through equilibrium, it becomes possible to also solve for *where* a force must act.

> ***The Process***
>
> **0.** Identify the Object!
>
> **1.** Identify the forces acting on the Object.
>
> **2.** Draw the Free-Body Diagram, clearly identifying the pivot.
>
> **3.** Separately, for each force acting on the Object:
>
> - draw the object and the coordinates
> - draw the force, placing it on the object where it is acting
> - draw the position vector $\vec{r}$ from the pivot to where the force is acting
> - determine the components of the force, and the contribution to torque.
>
> **4.** Use Newton's 1st Law to write the equations to be solved.
>
> **5.** Solve the equations for the unknown quantities.

2.4.0 Identify the Object!

Write it down: *what is the object?* Be very clear and specific about what is the Object. Only after you have specified the Object can you expect a clear answer to questions like "what is the force?"

With the ability to analyze rotation using torque we will now be able to consider more realistic problems of biomechanics that involve the human body. As we saw in the discussion about Newton's 3rd Law (1.3.1) when a person is the "cause" of the forces in the system it can become very easy to confuse effect and cause. Since our goal will be to determine what is happening ***to*** the person's body it is critical to not confuse that with what is being done ***by*** the person's body. Clearly identifying the Object will reduce the chance of confusing the two.

2.4.1 Identify the Forces

Please remember and try to appreciate the importance of this step: This is when you ***think physically*** about the situation. Take your time with this step and, using your choice from Step 0, be very clear about what force is associated with what interaction. Be careful with cause and effect, and use Newton's 3rd Law to reason about the direction of the force acting on the Object.

This is especially important in the biomechanical context, since it is very easy to confuse the action performed by a person (the force they exert on something else) with the reaction on them (the force that is acting on them). As we will typically be concerned with the forces acting inside the body, we will need to forces acting *on the person*, not they forces they are exerting on other things. Be very careful with this Step.

2.4.2 Draw the Free-Body Diagram

When considering torque, the Object can *not* be idealized as a point. Its geometry – specifically where on the object each force acts – is crucial. Draw a cartoon of the object, an outline showing its shape, and then place each force on the object where it is acting. Unlike what we did in section 1.4 the location of the application of the force on the Object is now *critical*.

Drawing a Free-Body Diagram (FBD)

2A. Draw a cartoon of the Object, with each of the forces identified in Step 1 drawn on the object where they act.

2B. Draw the "conventional FBD" with the forces each starting on a dot (which represents the object).

2C. Draw the sum of forces, in counter-clockwise order around the forces in the diagram from Step 2B (usually starting with gravity).

2D. Think Physically to correct the sums, if necessary: Adjust magnitudes based on the forces, *and on the torques*, summing to zero; Find friction; etc. Iterate between 2C and 2D until the sum is correct.

2E. Explain physically what the correct sums of forces and torques mean for the Object's static stability or trajectory.

As was the case in chapter 1 for forces, the "artistic quality" of the diagram is irrelevant. What matters is how the diagram can help you reason about the geometry of the forces and torques acting on the object.

It can not be over-emphasized that Step 2D, where we reason physically about the forces and torques, is where you should put the majority of your effort. As was the case in chapter 1 this is where adjustments of the magnitudes of the forces can be determined qualitatively by refining the sum of forces. But now we also have the additional information that can be determined by reasoning about how the torques sum to zero.

With the forces drawn on the object where they act we can *estimate* the relative amount of torque exerted by each force. This will often let us reason about the relative magnitudes of the forces. The rule of thumb "larger distance, smaller force, or smaller distance, larger force" can often lead us to correct conclusions about which the forces must be larger and which must be smaller. (This kind of reasoning feeds back into the iteration of the sum of forces being zero.)

2.4.3 Components

For each force acting on the object, make a separate drawing of the object *with the picot clearly identified*, and draw the force where it acts on the object. (Size is key, especially for finding angles and distances in determining contributions to torque.)

Find the component of forces, as before. But now take the additional step to find the contribution to torque from each force. The individual drawings of the forces acting on the object will help in correctly determine the distances and angles needed to find the torque.

Remember that each torque is relative to the pivot. Be careful with the geometry! Be clear about which method you are using to calculate the torque: line of action and moment arm ($\pm Fd$)?, or perpendicular component ($Fr\sin\theta$)? Have you been give, or can you easily identify, the moment arm? Where is the angle measured from? The size of your drawing will determine how easy it is to preform the correct constructions needed to determine the geometry.

2.4.4 Use Newton's 1st Law

When an object is in static equilibrium, it is neither moving nor rotating. For us, the statement of Newton's 1st Law has three equations:

$$\sum F_x = 0\,\mathrm{N} \tag{2.16}$$
$$\sum F_y = 0\,\mathrm{N} \tag{2.17}$$
$$\sum \tau_z = 0\,\mathrm{N{\cdot}m} \tag{2.18}$$

The equation for the sum of x-components of the forces says that the object moving neither left nor right. The equation for the sum of y-components of the forces says that the object moving neither up nor down. The equation for the sum of z-components of the torques says that the object not rotating about the chosen axis of rotation (the "pivot"). Together these are the mathematical statements saying that the object is in static equilibrium – it is not moving, and it does not start moving.

2.4.5 Solve for the Unknowns

Three equations allows for the determination of three unknowns. As we will see in the examples (and explore in the exercises) the work-flow will typically be like this:

- Use the sum of torques to solve for one unknown (typically an unknown magnitude of force, or an unknown distance to place where a force is being exerted);
- Then use the sum of forces to solve for the two components of the unknown force acting at the pivot.

Since the force at the pivot does not contribute a torque, the components (or magnitude) of that unknown force does not appear in the sum of torques equation. This places the unknowns into separate equations, and the algebra is usually quite simple. (Typically it is the geometry of determining the torques that is much more difficult than the solving for the unknowns.)

The one aspect to be *very careful* of is the ***units*** of the various quantities that you are substituting into the equations before solving. It is recommended that you explicitly and carefully fully substitute and write-out all the quantities with their units in the equations *before* you reach for your calculator. This is the place to notice if the units of your "result" actually match what you are calculating. For example, if you forgot to multiply a mass by "g", then you might find that what you know should be a distance instead has units of N·m/kg (which is non-sense). That's when you go back and check your previous steps.

2.5 Examples

These examples will fall into three categories.

The first are those where the Object is not a person. These are categorized as being *mechanical*. They will generally be geometrically simple solid objects.

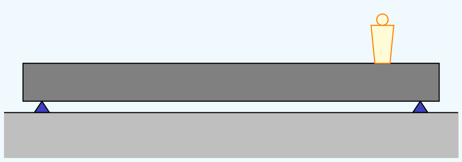
The second and third categories are those in which the Object *is* a person. The second category are those where the person *as a whole* is considered as the Object. In those cases the feature to study carefully is the relation between what the person is doing (their action) and what is being done to them (the reaction).

The third category are those in which the Object is *a portion* of a person's anatomy. In this category the goal will be to determine the forces that are acting *internally*, typically the tension in a muscle, and/or the forces of contact and attachment between skeletal segments. These examples will build upon what we learn in the second category: Starting from what the person is doing, we can determine the reaction force(s) on them, and then find the forces that consequently must be acting inside the person's body. The study of these situations is the goal of the biomechanics segment of this course.

2.5.1 Mechanical examples

Example 2.4 : Supporting a Stage with a Person on it

A performer is standing on a stage. The stage is supported at each end by some metal brackets. (The shape of the supports is not important.) The stage itself has a mass of $m_{\text{stage}} = 1200\,\text{kg}$, and its center of mass is at its geometric center. The stage is 11.00 m wide and each support is 0.50 m from the end. The performer is standing 1.50 m from the right end of the stage, and their mass is $m_{\text{person}} = 65.00\,\text{kg}$. Find the forces exerted on the stage by each of the supports.



We are asked to find two forces, the forces exerted by the supports. So we will need to answer with two force vectors. Since the person is standing closer to the support on the right we expect that the support on the right will have to exert a greater force than the support at the other end.

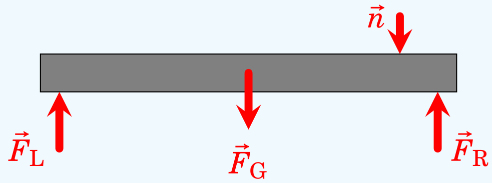
Step 0: Since we are asked to find the forces exerted on the stage, the stage is the object.

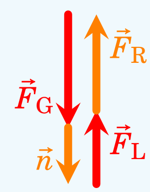
Step 1: The stage has a large mass, and gravity will be pulling downwards on it. Even if the performer was not there, the supports would have to support against that weight. Each support will have to exert an upwards force on the stage to support it.

The person is touching the stage, so they exert a normal on the stage. The force that the person exerts is *not* gravity! Think for a moment of the *person* as the object: Gravity pulls down on them, and the stage exerts an upwards normal. By Newton's 3rd Law the person exerts the opposite of this normal on the stage. This is the simplest case, where the normal happens to have a magnitude equal to $m_{\text{person}}g$, but the normal is not itself a gravitational force.

With the stage as the object, with this normal force being exerted on it from above, the supports will have to provide greater forces than just supporting the stage's own weight.

Step 2: Below are the FBD and the check of sum of forces:

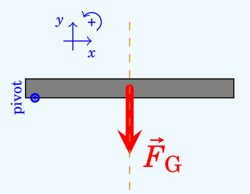


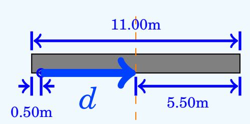


This check of the sum of the forces is only qualitative because the force of gravity is, in reality, about twenty times bigger than the normal exerted by the person.

We also expect that the force of support on the right will have to be slightly greater than that on the left because the force exerted by the person is closer to the right.

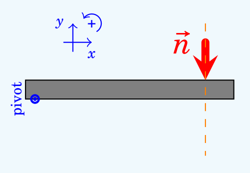
Step 3: For each force acting on the object we now need to find its components and the contribution it makes to the torque. In the solution presented here this work seems to take a lot of space. Do not be alarmed! This is only because I am explaining each step in great detail that it appears long. When you get to doing problems of this type you will find how compactly these steps can be done.

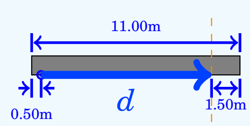




The components of the gravitational force are $F_{\text{G},x} = 0\,\text{N}$ and $F_{\text{G},y} = -m_{\text{stage}}\,g = -(1200\,\text{kg})(9.81\,\text{N/kg}) = -11\,772\,\text{N}$. This force acts at the center of the stage, which is $\frac{1}{2} \times 11.00$ m $= 5.50$ m from the left end. The pivot is 0.50 m from the left end. So the distance from the pivot to the line of action is 5.00 m.

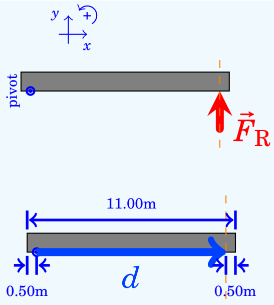
This force, by itself, would make the object begin to rotate clockwise (which is negative, relative to our choice of coordinates). Thus the contribution to torque from this force is $\tau_{\text{G},z} = -(11\,772\,\text{N})(5.00\,\text{m}) = -58\,860\,\text{N}{\cdot}\text{m}$.





The components of the normal that the person exerts on the object are $n_x = 0\,\text{N}$ and $n_y = -m_{\text{person}}\,g = -(65.00\,\text{kg})(9.81\,\text{N/kg}) = -637.65\,\text{N}$. This force acts 1.50 m from the from the right end of the object. The pivot is 0.50 m from the left end. Since the object (the stage) is 11.00 m wide the distance from the pivot to the line of action is 9.00 m.

This force, by itself, would also make the object begin to rotate clockwise. Thus the contribution to torque from this force is $\tau_{n,z} = -(637.65\,\text{N})(9.00\,\text{m}) = -5\,738.85\,\text{N}{\cdot}\text{m}$.



The components of the force of the support on the right are $F_{R,x} = 0\,N$ and $F_{R,y} = +F_R$. This force acts 0.50 m from the from the right end of the object. The pivot is 0.50 m from the left end. Since the object (the stage) is 11.00 m wide the distance from the pivot to the line of action is 10.00 m.

This force, by itself, would also make the object begin to rotate counter-clockwise. Thus the contribution to torque from this force is $\tau_{R,z} = +F_R \times 10.00\,m$.

Lastly, the force of support at the left end has components $F_{L,x} = 0\,N$ and $F_{R,y} = +F_R$, and contributes no torque ($\tau_{L,z} = 0\,N{\cdot}m$) because it acts *at* the pivot.

Step 4: The Newton's 1st Law for the forces applied is

$$\vec{F}_G + \vec{n} + \vec{F}_L + \vec{F}_R = \vec{0}\,N \tag{2.19}$$

The x-components of all the forces are zero. Consequently the x-component of the sum contains no information for us. The y-component equation is

$$F_{G,y} + n_y + F_{L,y} + F_{R,y} = 0\,N \tag{2.20}$$

$$-11\,772\,N - 637.65\,N + F_L + F_R = 0\,N \tag{2.21}$$

If we pause here we can see why we need to consider torques. The sum of forces, in this situation, only provides one equation for two unknowns. The sum of forces alone is not sufficient to determine the conditions for static equilibrium. The sum of torques being zero provides an addition equation that produces a solution.

The Newton's 1st Law for the torques applied is

$$\tau_{G,z} + \tau_{n,z} + \tau_{L,z} + \tau_{R,z} = 0\,N{\cdot}m \tag{2.22}$$

$$-58\,860\,N{\cdot}m - 5\,738.85\,N{\cdot}m + 0\,N{\cdot}m + F_R \times 10.00\,m = 0\,N{\cdot}m \tag{2.23}$$

Now we have two equations (the x-component of the sum of forces, and the z-component of the sum of torques) for the two unknowns: the magnitudes of the forces acting at the supports.

Step 5: The equation for torque has only one unknown in it, F_R. Solving that equation gives us

$$F_R = \frac{58\,860\,N{\cdot}m + 5\,738.85\,N{\cdot}m}{10.00\,m} = +6\,460\,N \tag{2.24}$$

This we then substitute into the equation for the y-components of the forces:

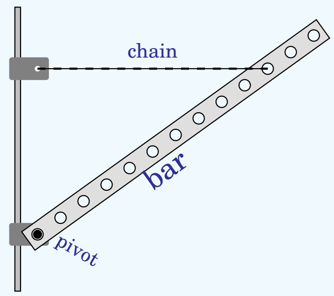
$$0\,N = -11\,772\,N - 637.65\,N + F_L + F_R \tag{2.25}$$

$$F_L = (+11\,772 + 637.65 - 6\,460)\,N = +5\,950\,N \tag{2.26}$$

Both of these results are for the magnitudes of the forces. The fact that they are both positive indicates that our choices of direction for each of the forces were correct.

Answer: The force exerted by the support on the right is 6 460 N upwards. The force exerted by the support on the left is 5 950 N, upwards. These are the two force vectors were we asked to find. As expected, the support on the right is supporting more of the performer's weight and so is greater in magnitude than that exerted by the support on the left.

Example 2.5 : Equilibrium Lab



A thick, flat metal bar is suspended as shown in the diagram: pivoted at one end, and supported at the other by a horizontal chain. The chain is 76.5 cm long, and is 55.0 cm above the pivot.

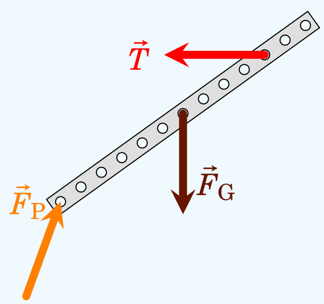
The bar has several holes drilled through it. The distance between the first (the pivot) and the one where the chain is attached is 94.8 cm. The bar has mass 1.170 kg, and its center of mass is 55.0 cm from the pivot.

What is the tension in the chain?

Step 0: The object is the metal bar.

Step 1: The metal bar is interacting with the Earth (gravity, vertically downwards), the support (a contact force at the pivot), and the chain that is holding it in equilibrium (tension, horizontally towards the left). If we are able to solve the problem using the sum of torques about the pivot, the force at the pivot will be irrelevant since it does not contribute a torque.

Step 2:

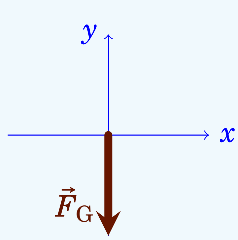


With gravity pointing downwards and tension pulling towards the left, the force of support at the pivot must point up and towards the right.

It is difficult to judge exactly, but it looks like the moment arm of the tension is greater than the moment arm of gravity, so the tension might be slightly less than gravity.

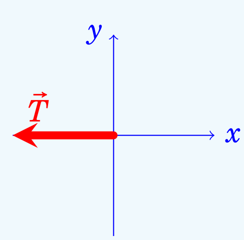
Step 3: The components of the forces are easy to state, since gravity is vertical and the tension is only horizontal. But finding their contributions to torque is a little trickier since the contribution from gravity requires solving a problem of the geometry to find either the moment arm, or the angle between $\vec{F}_G$ and its corresponding $\vec{r}$.

The components of the forces are



$F_{G,x} = 0\,\mathrm{N}$

$F_{G,y} = -(1.170\,\mathrm{kg})(9.81\,\frac{\mathrm{N}}{\mathrm{kg}}) = -11.48\,\mathrm{N}$

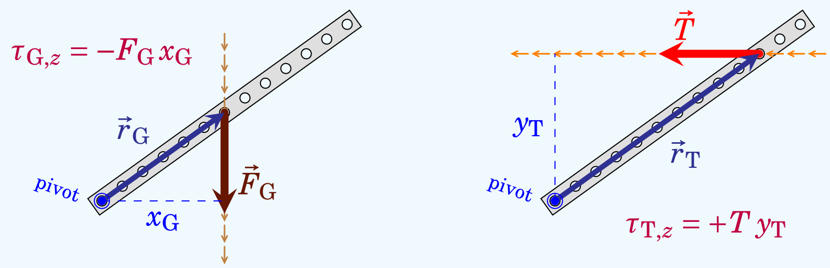


$T_x = -T$

$T_y = 0\,\mathrm{N}$

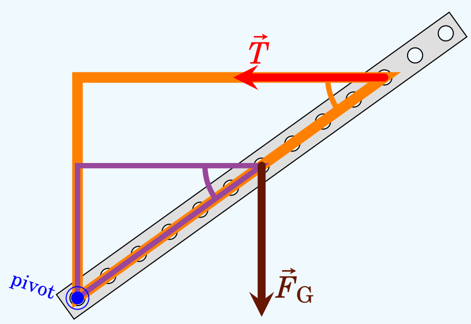
Both components ($F_{P,x}$ and $F_{P,y}$) of the force of support at the pivot are unknown (although we can anticipate that they will both have to be positive).

The force of support at the pivot applies no torque about the pivot. The geometry for the two non-zero contributions to torque are shown below.



We have the moment arm y_T for the tension, but we do not have the moment arm x_G for gravity.

We know all three sides of the right-triangle that has r_T as its hypotenuse. The unknown length x_G that we need is one side of a right-triangle of which whose hypotenuse ($r_G = 55.0\,\text{cm}$) is the only value we have. To find x_G we note that those two triangles are *similar*; they have identical angles in their corners, as we can see in the diagram below.



For the larger (orange) triangle, the cosine of the angle in its top right corner equals the ratio x_T/r_T. In the smaller (purple) triangle it has the same angle in its top right corner, for which its cosine is the ratio x_G/r_G. Since the angles are the same, these ratios are the same, and we can write

$$\frac{x_G}{r_G} = \frac{x_T}{r_T} \tag{2.27}$$

$$x_G = r_G \times \frac{x_T}{r_T} = (55.0\,\text{cm})\frac{76.5\,\text{cm}}{94.8\,\text{cm}} = 44.38\,\text{cm} \tag{2.28}$$

We now have sufficient data to write the contributions to torque:

$$\tau_{G,z} = -F_G x_G = -(11.48\,\text{N})(0.4438\,\text{m}) = -5.094\,\text{N·m} \tag{2.29}$$

$$\tau_{T,z} = +T y_T = +T \times 0.550\,\text{m} \tag{2.30}$$

$$\tau_{P,z} = 0\,\text{N·m} \tag{2.31}$$

Step 4: If we write the sum of forces we would be able to solve for the force of support at the pivot, but we were not asked for that. To solve for the tension in the chain we need only consider the sum of torques being zero about the pivot:

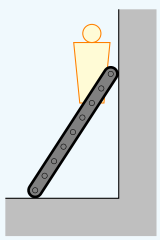
$$\tau_{G,z} + \tau_{T,z} + \tau_{P,z} = 0\,\mathrm{N{\cdot}m} \quad (2.32)$$
$$(-5.094\,\mathrm{N{\cdot}m}) + (+T \times 0.550\,\mathrm{m}) + (0\,\mathrm{N{\cdot}m}) = 0\,\mathrm{N{\cdot}m} \quad (2.33)$$

Step 5: Solving that equation for the unknown tension gives

$$(-5.09\,\mathrm{N{\cdot}m}) + (+T \times 0.550\,\mathrm{m}) + (0\,\mathrm{N{\cdot}m}) = 0\,\mathrm{N{\cdot}m} \quad (2.34)$$
$$T = +5.094\,\mathrm{N{\cdot}m}/0.550\,\mathrm{m} = 9.26\,\mathrm{N} \quad (2.35)$$

Answer: The tension in the chain required to maintain static equilibrium is $T = 9.26\,\mathrm{N}$ which, as was guessed, is slightly smaller than the magnitude of gravity.

Example 2.6 : Ladder against a wall



A person is standing on a ladder that is leaning against a wall. The ladder is 4 m long. The angle between the ground and the ladder is 57°. The person has a weight of 700 N and is standing 1 m from the top end of the ladder.

Ignore the weight of the ladder (it's a very light aluminum ladder). There is friction between the ladder and the ground, but there is no friction between the ladder and the wall. (This is made possible by small wheels at the top end of the ladder which allow you to slide the end of the ladder up the wall when putting the ladder in place.)

Find: **1.** The magnitude of the normal acting on the ladder at the wall; **2.** The magnitude of the normal acting on the ladder at the ground; **3.** The magnitude and direction of the friction acting on the ladder at the ground.

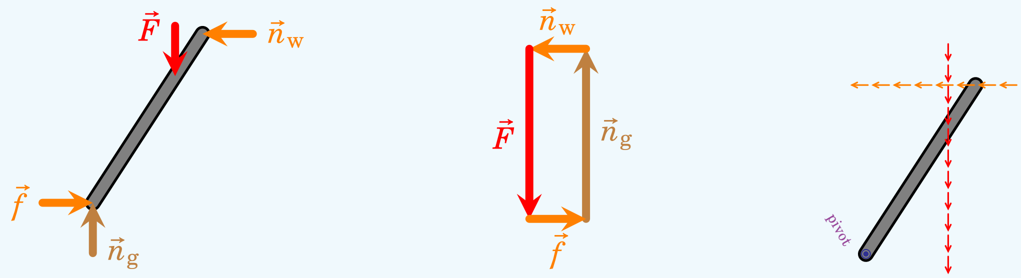
Step 0: In the statement of the problem we are asked to find the forces acting on the ladder. This means that the ladder is our choice of Object.

Step 1: The interaction with the Earth is always present, but we are told to ignore this force because it will be small. (We recall that have done this before in contexts like pulleys, where a particular force is small in comparison to all other forces acting, and that neglecting it will not change the results by a significant amount.)

The person standing on the ladder is exerting a force on the ladder. To understand this contribution we can think about the action-reaction (Newton's 3rd Law) interaction between the person and the ladder. If, for just this moment, we consider the person as the Object, then we find that gravity is pulling downwards and there is a normal from the ladder acting upwards. With the upwards normal on the person as the "action", there is a downwards force on the ladder as the "reaction". In the context of the problem, with the ladder as the Object, it is this downwards force that we need to include.

The ladder makes contact with the surface of the wall and the surface that is the ground. There will be *two* normal forces, one at each surface. We are told explicitly that there is no friction at the wall, and that we are to find the friction at the ground.

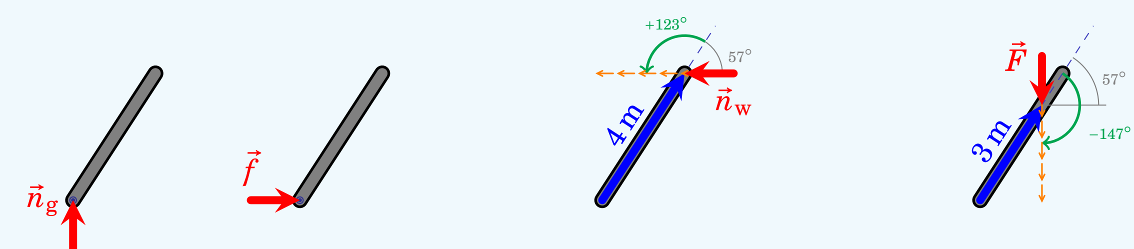
Step 2: The Free-Body Diagram for the ladder, and the check of sum of forces, is:



To get a qualitative estimate of the size of the forces acting on the ladder we need to think about the sum of torques around the bottom end of the ladder. (Here we *choose* that point as the "pivot".) Specifically, where are the lines of action for the normal from the wall and the force exerted by the person? (Diagram above on the far right.) The distances from those lines to the pivot determine their relative magnitude: For these torques to balance the force along the line that is further from the pivot must be smaller than the force along the line that is closer to the pivot. Looking at the diagram we can see this means that the normal from the wall must be smaller than the force applied by the person. (We use this information to construct our check of the sum of forces.)

Lastly, the check of the sum of forces shows that the friction acting at the ground must point towards the wall. This is because the only other force acting horizontally is the normal from the wall, which acts towards the left.

Step 3: With the exception of the person's weight, we know none of the magnitudes of the forces. We do, however, know all their directions. The components of the forces and their contributions to the torque about the pivot are:



$$n_{{\rm g},x} = 0\,{\rm N} \qquad n_{{\rm g},y} = +n_{\rm g} \qquad \tau_{ng,z} = 0\,{\rm N{\cdot}m}$$

$$f_x = f_x \qquad f_y = 0\,{\rm N} \qquad \tau_{f,z} = 0\,{\rm N{\cdot}m}$$

$$n_{{\rm w},x} = -n_{\rm w} \qquad n_{{\rm w},y} = 0\,{\rm N} \qquad \tau_{n{\rm w},z} = (n_{\rm w})(4\,{\rm m})\sin(+123^\circ) = +(n_{\rm w})(3.355\,{\rm m})$$

$$F_x = 0\,{\rm N} \qquad F_y = -700\,{\rm N} \qquad \tau_{F,z} = (700\,{\rm N})(3\,{\rm m})\sin(-147^\circ) = -1144\,{\rm N{\cdot}m}$$

Note that in the diagram for the friction at the ground we have drawn friction acting towards the right. But we have not yet determined that this correct, quantitatively. For this reason we leave f_x as the unknown to be solved for.

Step 4: The equations for static equilibrium are as follows: The sum of the horizontal

(x-components) of the forces sum to zero

$$F_x + f_x + n_{g,x} + n_{w,x} = 0\,\mathrm{N} \tag{2.36}$$
$$0\,\mathrm{N} + f_x + 0\,\mathrm{N} - n_w = 0\,\mathrm{N} \tag{2.37}$$

The sum of the vertical (y-components) of the forces sum to zero

$$F_y + f_y + n_{g,y} + n_{w,y} = 0\,\mathrm{N} \tag{2.38}$$
$$-700\,\mathrm{N} + 0\,\mathrm{N} + n_g + 0\,\mathrm{N} = 0\,\mathrm{N} \tag{2.39}$$

The sum of the (z-components of the) torques about the end of the ladder at the ground sum to zero

$$\tau_{F,z} + \tau_{F,z} + \tau_{ng,x} + \tau_{nw,x} = 0\,\mathrm{N{\cdot}m} \tag{2.40}$$
$$-1144\,\mathrm{N{\cdot}m} + 0\,\mathrm{N{\cdot}m} + 0\,\mathrm{N{\cdot}m} + (n_w)(3.355\,\mathrm{m}) = 0\,\mathrm{N{\cdot}m} \tag{2.41}$$

Step 5: The equation for the y-components of force lets us solve for the normal from the ground:

$$-700\,\mathrm{N} + n_g = 0\,\mathrm{N} \tag{2.42}$$
$$n_g = +700\,\mathrm{N} \tag{2.43}$$

Since there was no friction at the wall to contribute a vertical force, it makes sense that the normal at the ground has to support the full weight being applied to the ladder. (Remember that we were also told to ignore any weight of the ladder itself.)

The equation for the sum of torques lets us solve for the normal from the wall:

$$-1144\,\mathrm{N{\cdot}m} + (n_w)(3.355\,\mathrm{m}) = 0\,\mathrm{N{\cdot}m} \tag{2.44}$$
$$n_w = \frac{+1144\,\mathrm{N{\cdot}m}}{3.355\,\mathrm{m}} = +341\,\mathrm{N} \tag{2.45}$$

In our analysis in Step 2 we reasoned that the normal from the wall would have to be smaller that the force due to the person (since its line of action is further from the "pivot"). This result agrees with that reasoning.

Finally, with the normal from the wall we can use the equation for the x-components of force to solve for the friction:

$$+f_x - n_w = 0\,\mathrm{N} \tag{2.46}$$
$$f_x = +n_w = +341\,\mathrm{N} \tag{2.47}$$

As deduced through our construction of the check of the sum of forces, the friction has the same magnitude as the normal from the wall. That this result is positive means that our choice for its direction (pointed towards the right) was correct.

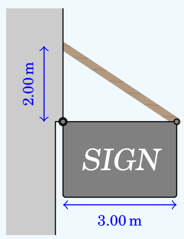
Answer: We were asked to find three things:

1. The magnitude of the normal acting on the ladder at the wall: 341 N.

2. The magnitude of the normal acting on the ladder at the ground: 700 N.

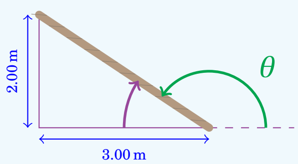
3. The magnitude and direction of the friction acting on the ladder at the ground. Friction has a magnitude of 341 N and is pointed towards the right.

Example 2.7 : Cable holding a Sign



A sign of mass 312 kg is attached to the side of a building, as shown in the diagram. Find the tension in the cable. Then find the magnitude and direction of the force at the pivot. (Hint: Before calculating any other quantities first find the angle between the cable and the horizontal.)

We are asked to find the tension in the cable (which is a magnitude), and the force at the pivot (which is a vector). Our answer for the force at the pivot is to be expressed in terms of its magnitude and direction. Before we work towards finding any of those we are told to calculate the angle between the cable and the horizontal. Here's the diagram:

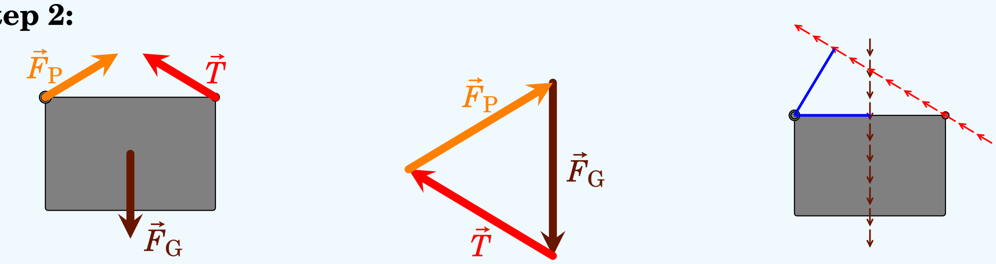


Using the triangle on the left with the cable as its hypotenuse, we calculate with the inverse tangent $\tan^{-1}(2.00\,\text{m}/3.00\,\text{m}) = +33.7°$. But the angle that we want is $\theta = 180° - 33.7° = 149°$.

Step 0: The Object is the sign.

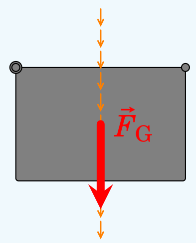
Step 1: The sign is interacting with the Earth (by gravity), and the building (by the cable, and at the pivot).

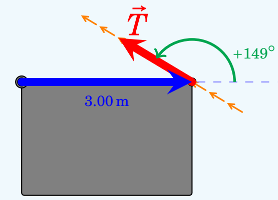
Step 2:



When we first draw the free-Body Diagram we do not know the direction of the force at the pivot. If we then draw the check of the sum of forces we can see that the force at the pivot must point towards the right, but we don't know its vertical component. If we then sketch the line of action for gravity and for the tension, it looks like the moment arm (the distance d) for both of those torques might be almost the same. So at this level of qualitative reasoning we can not be certain which way the force at the pivot points.

Step 3: The components of the forces and the contributions to torque about the pivot are:





$F_{G,x} = 0\,\text{N}$

$F_{G,y} = -mg = -(312\,\text{kg})(9.81\,\text{N/kg}) = -3061\,\text{N}$

$\tau_{G,z} = -(3061\,\text{N})(1.50\,\text{m}) = -4591\,\text{N·m}$

$T_x = T\cos 149°$

$T_y = T\sin 149°$

$\tau_{T,z} = (T)(3.00\,\text{m})\sin(+149°) = +T \times 1.545\,\text{m}$

The force at the pivot is unknown, so we leave $F_{P,x}$ and $F_{P,y}$ as unknowns for which we must solve. However, that force is *at* the pivot, so we do know that it does not exert a torque: $\tau_{P,z} = 0\,\text{N·m}$.

Step 4: The equations for static equilibrium are as follows: The sum of the horizontal (x-components) of the forces sum to zero

$$F_{G,x} + T_x + F_{P,x} = 0\,\text{N} \tag{2.48}$$
$$0\,\text{N} + T\cos 149° + F_{P,x} = 0\,\text{N} \tag{2.49}$$

The sum of the vertical (y-components) of the forces sum to zero

$$F_{G,y} + T_y + F_{P,y} = 0\,\text{N} \tag{2.50}$$
$$-3061\,\text{N} + T\sin 149° + F_{P,y} = 0\,\text{N} \tag{2.51}$$

The sum of the (z-components of the) torques about the end of the ladder at the ground sum to zero

$$\tau_{G,z} + \tau_{T,z} + \tau_{P,z} = 0\,\text{N·m} \tag{2.52}$$
$$-4591\,\text{N·m} + T \times 1.545\,\text{m} + 0\,\text{N·m} = 0\,\text{N·m} \tag{2.53}$$

Step 5: Solving the torque equation for the unknown tension, we get

$$-4591\,\text{N·m} + T \times 1.545\,\text{m} = 0\,\text{N·m} \tag{2.54}$$
$$T = \frac{+4591\,\text{N·m}}{1.545\,\text{m}} = +2971\,\text{N·m} \tag{2.55}$$

Comparing this with the magnitude of the sign's weight (3061 N) we find that they are almost the same. This agrees with our guess made while constructing the check of sum of forces, where the moment arm (the distance d) looked like them might be the same.

Substituting this value for the tension into the equation for the x-component of force, we get

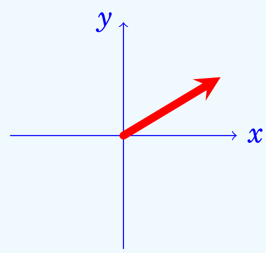
$$T\cos 149° + F_{P,x} = 0\,\text{N} \tag{2.56}$$
$$F_{P,x} = -(2971\,\text{N·m})\cos 149° = +2546\,\text{N} \tag{2.57}$$

Similarly, from the equation for the y-component of force, we get

$$-3061\,\mathrm{N} + T\sin 149^\circ + F_{\mathrm{P},y} = 0\,\mathrm{N} \quad (2.58)$$

$$F_{\mathrm{P},y} = +3061\,\mathrm{N} - (2971\,\mathrm{N{\cdot}m})\sin 149^\circ = +1531\,\mathrm{N} \quad (2.59)$$

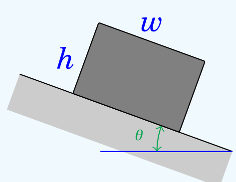
We were asked to express the force at the pivot in terms of its magnitude and direction.



From its components found above, we see that $\vec{F}_\mathrm{P}$ points into the first quadrant. Calculating with the inverse tangent $\tan^{-1}\left(\frac{+1531\,\mathrm{N}}{2546\,\mathrm{N}}\right) = +31^\circ$, which is in the correct quadrant. The magnitude is $F_\mathrm{P} = \sqrt{(2546\,\mathrm{N})^2 + (1531\,\mathrm{N})^2} = 2971\,\mathrm{N}$. We note that this magnitude is close to that of the tension in the cable.

Answer: The tension in the cable is 2546 N. The force at the pivot is 2971 N, pointed towards the right at 31° above the horizontal.

Example 2.8 : (Advanced) Where the Normal acts



A box of weight mg is at rest on surface that is inclined at an angle θ, as shown in the diagram. It has height h and width w. **1.** Find the magnitude of the normal and the required friction for the box to be in static equilibrium (sum of forces zero). **2.** Find the effective location of where the normal force acts for the box to be in static equilibrium (sum of torques zero).

We are asked to first solve for the forces required for the box to be in equilibrium. We know that the normal will be perpendicular to the surface, pointed at the object, but with a magnitude less than mg since friction will assist in supporting the object. We know that friction will be parallel to the surface, and we reason that it should point *upwards* along the incline. These will be found by requiring that the forces on the object sum to zero.

After that we are to solve for *where* on the object the normal acts by applying the constraint that the sum of torques is zero. Since there is no physical hinge or joint, we will be free to choose the location of the axis of rotation (the "pivot"). The math that we have to solve will depend upon that choice, by the physical result we obtain will be independent of that choice.

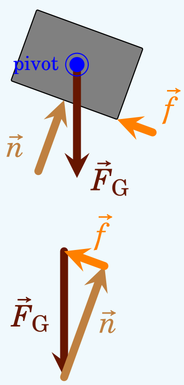
Step 0: The object is the Box.

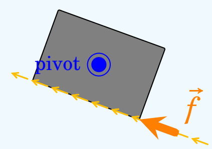
Step 1: The object is interacting with the Earth, and the surface. Consequently the forces acting on the object are gravity, the normal, and friction.

Step 2:

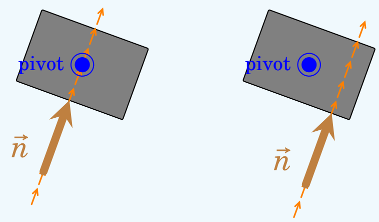
The object is easy to draw, but we must make a choice of where the "pivot" will be since there is no actual physical pivot (like a hinge) constraining the object. Obvious choices might be one of the corners of the box, or its center of mass. For this example I will choose the center of mass as the pivot. Requiring that the sum of torques about this pivot be zero is saying that the object does not tumble (roll end-over-end) down the incline.

(For the sake of completeness I will comment that choosing one of the corners as the pivot would also lead to the same results we find below, but the trigonometry and algebra we would have to solve becomes much more complicated – so we won't do that here.)





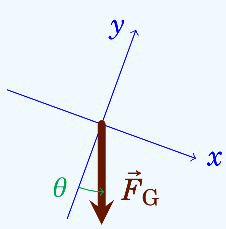
The line of action of friction is along the bottom surface of the object, independent of the incline of the surface. The friction points upwards along the incline, exerting a negative torque about the center of mass.



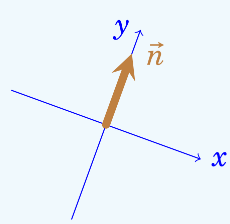
If the normal remains at the center of the object, its line of action will continue to pass through the center of mass, and will provide no torque. To counter the negative torque exerted by friction the normal's line of action must shift down the incline, as shown, to produce a positive torque.

Step 3:

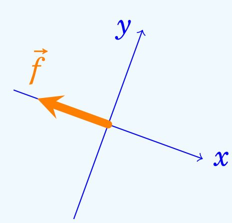
A common approach to solving objects on inclines is to choose a tiled coordinate system, with one axis parallel to the surface and the other perpendicular to the surface. With that choice the normal and friction forces are aligned with separate axes: the friction along the axis parallel to the surface, and the normal along the axis perpendicular to the surface. Consequently the unknown magnitudes of those two forces appear separate from each other in the equations for the sums of the components of the forces. With the unknowns already isolated, no complicated algebra is required to solve for their values.



$F_{G,x} = +mg\sin\theta$
$F_{G,y} = -mg\cos\theta$



$n_x = 0\,N$
$n_y = +n$

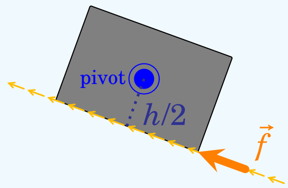


$f_x = -f$
$f_y = 0\,N$

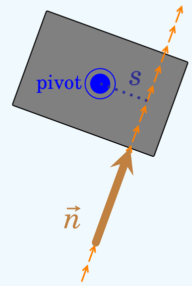
Notice that the components of the gravitational force do not follow the usual pattern of "cos for x and sin for y" because of the way that the angle θ is defined.

The torques exerted about the center of mass by the forces are determined below. In what follows the distance s is the unknown value we want to find: it tells us where the normal (modelled as a single force) acts to maintain static equilibrium.

$\tau_{f,z} = -f\,h/2$



$\tau_{n,z} = +n\,s$



By definition gravity does not exert a torque about the center of mass ($\tau_{G,z} = 0\,\mathrm{N{\cdot}m}$).

Step 4: The sum of forces must be zero:

$$\vec{F}_G + \vec{n} + \vec{f} = \vec{0}\,\mathrm{N} \tag{2.60}$$

The components of the sum that are parallel to the incline are

$$F_{G,x} + n_x + f_x = 0\,\mathrm{N} \tag{2.61}$$
$$(+mg\,\sin\theta) + 0\,\mathrm{N} + (-f) = 0\,\mathrm{N} \tag{2.62}$$

The components of the sum that are perpendicular to the incline are

$$F_{G,y} + n_y + f_y = 0\,\mathrm{N} \tag{2.63}$$
$$(+mg\,\cos\theta) + (+n) + 0\,\mathrm{N} = 0\,\mathrm{N} \tag{2.64}$$

The sum of torques about the center of mass are:

$$\tau_{G,z} + \tau_{n,z} + \tau_{f,z} = 0\,\mathrm{N{\cdot}m} \tag{2.65}$$
$$0\,\mathrm{N{\cdot}m} + (+n\,s) + (-f\,h/2) = 0\,\mathrm{N{\cdot}m} \tag{2.66}$$

Step 5: Solving the sum of forces gives

$$f = mg\,\sin\theta \tag{2.67}$$
$$n = mg\,\cos\theta \tag{2.68}$$

Substituting that into the equation for the sum of torques gives

$$(+n\,s) + (-f\,h/2) = 0\,\mathrm{N{\cdot}m} \tag{2.69}$$
$$+mg\,\cos(\theta)\,s - mg\,\sin(\theta)\,h/2 = 0\,\mathrm{N{\cdot}m} \tag{2.70}$$
$$s = \tfrac{1}{2}h\,\tan\theta \tag{2.71}$$

Answer:

The normal acts at a distance $s = \frac{1}{2}h\,\tan\theta$ from the center of the box. (It is geometrically challenging to prove, but this is vertically below the center of mass!)

Notice that the normal can only act on the surface of contact. This means that $s \le w/2$. For this reason

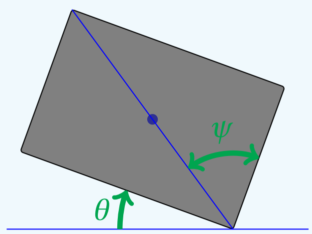
$$\tfrac{1}{2}w \ge s = \tfrac{1}{2}h\tan\theta \tag{2.72}$$

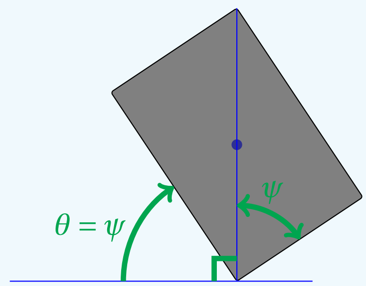
$$(w/h) \ge \tan\theta \tag{2.73}$$

$$\theta \le \tan^{-1}(w/h) \tag{2.74}$$

What does this mean, geometrically?

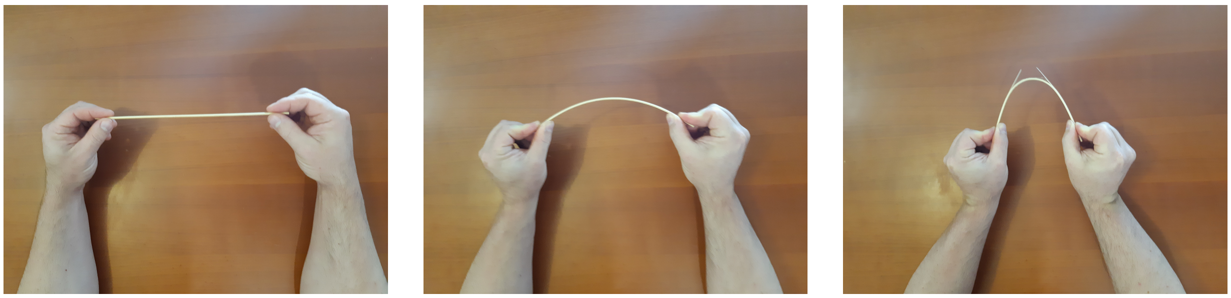
If we define $\psi = \tan^{-1}(w/h)$, then we notice that $\theta = \psi$ would have the center of mass vertically above the corner of the box (shown below). Tilting the incline any further would place the center of mass beyond the supporting surface of contact, and the box would begin to tumble down the incline.





Chapter 3

Materials



Experience teaches us that not all things are perfectly rigid. Even materials like wood and metal, when they are thin enough, can be seen to bend when we push or pull on them. Think of a wooden chop stick, or a metal paperclip.

If we look at a thicker, more solid piece of material, it is not too difficult to imagine that we might be able to make it bend too, if we were strong enough. You can bend a chop stick with your hands, but you can't bend a door or large table. And you certainly could not bend the trunk of a large tree. You can bend a paper clip, but you can't bend a car door. And you certainly could not bend a steel beam from a building.

However, there was a machine that bent the car door into its shape from a flat sheet of metal. And there are (unfortunately) earthquakes that could bend the metal beams of buildings. The point of this line of thinking is to realize that there is no size of metal object that can *not* be bent, it's just that a proportionally larger force would be required.

This result is an example of a physical fact:

Everything bends.

Wood, plastics, metal; these we can all imagine bending. But glass? concrete? bones?!? Do these materials bend? Yes, they do, and it is possible to measure how much they bend. Stiffer materials bend less.

Experience also teaches us that if you bend something too far, it will break. Because the chemical bonds that hold the atoms of an object together represents only a finite amount of energy, it is possible to break any chemical bond. This underlies a critical physical fact:

Everything breaks.

In the context of your Physiotherapy Technology program the patients you will be helping will often have suffered from some injury where a part of their body, muscle, tendon, ligament, or bone, has broken. Being able to understand how parts of the body may break is the first step in understanding a path to restoration.

The physical properties of stiffness and strength are the subject of this chapter.

3.1 Everything Bends

Everything is made of atoms, and atoms are bound to each other by chemical bonds. It is through their chemical bonds that atoms exert the forces that hold the material together. Each chemical bond between the atoms in a material has an equilibrium length (the distance between the centers of the bound atoms). When an external force is applied to the object the length of the bonds change until a new equilibrium is achieved between the externally applied force and the bond forces between the atoms. In this section we will study the relationship between the forces applied to the object and the change in an object's geometry.

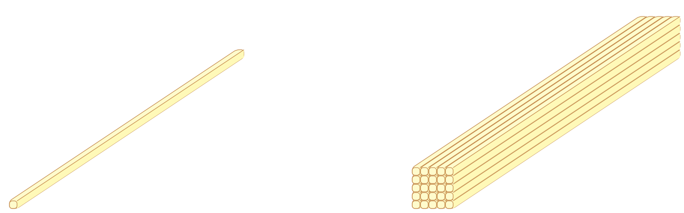
3.1.1 Stress

When an external force is applied to the object the length of the bonds change until a new equilibrium is achieved between the externally applied force and the bond forces between the atoms. The change in the length of an elastic band is an example of this.

Go get yourself a couple of identical elastic bands (three will do). Hold your hands up with your palms facing you. Now stretch a single elastic band between the little finger on each of your hands. Feel the force that takes. If we wrap three rubber bands around your fingers it becomes three times more difficult to stretch. But note! We haven't changed the material, just its geometry. The distance between your fingers when the elastics are not stretched has not changed. But the elastic material between your fingers is now three times thicker. The "strength" of a rubber band depends upon the cross-sectional area of the material.

So, is there a way to speak of the "strength" of a material independent of its geometry? Of course! That is what this section is about. Consider the following example.

A table leg made from a solid piece of wood can hold the same weight as a table leg made from a bunch of chopsticks bundled together. We think of a solid chunk of wood as being "stronger" than a thin chopstick. But both are made of the same material, and it is natural to think that their material has the same strength. What makes a solid table leg "stronger" than a chopstick is that it is wider. So our sense of "strength" is a blend of the actual stiffness of the material itself and the cross-sectional area of the object made from that material.



If we pull on both ends of a single chopstick, it may stretch a certain (small) amount. The amount it stretches will depend upon the amount of force we apply. If we pull on both ends of a piece of wood – imagined as a bundle of chopsticks – and we want it to stretch as much as the single chopstick by itself, then we will have to apply that original amount of force once *for each chopstick in the bundle*. If the chopsticks in the bundle are understood as the fibers of the solid piece of wood, then what matters is the "force per fiber" applied to the material. This means that the total amount of force required to stretch an object a specified amount is proportional to its cross-sectional area.

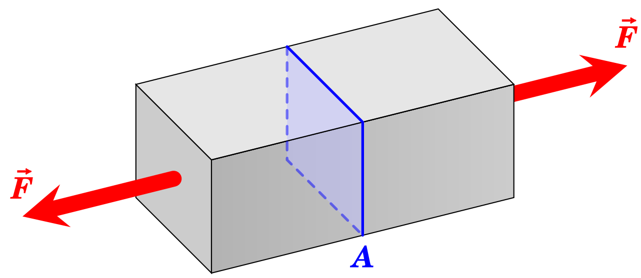
For this reason when we study the effect that an applied force has on the shape of an object, we measure the applied force per area. This ratio is called the ***stress***:

$$\text{stress} = \frac{\text{force}}{\text{area}} \tag{3.1}$$

The symbol used to denote stress is the Greek letter sigma (σ), and the mathematical statement that defines stress is written

$$\sigma = F/A \tag{3.2}$$

This equation says that the *stress* equals the magnitude of the applied force divided by the area that it is being distributed across.



Force per area is usually pressure, which is how our *pushing* on a surface is distributed. But stress is defined even when we *pull* on an object. We first saw the concept of pressure in chapter 1, sub-section 1.2.2, where we studied the details of the normal force at contact. We will return to study the concept of pressure in greater detail in chapter 4. When we study sound waves in chapter 5 air pressure will be one of the key physical quantities involved in its description. Pressure, in the contexts of the normal and "air pressure" measures how a push is distributed across an area. But stress is defined so that we can measure how a force is distributed across an area when we push, when we pull, apply a shear, or even twist on that area.

The units of stress are newtons per square metre. One newton per square metre is defined to be one ***pascal*** of pressure ($1\,\text{N/m}^2 = 1\,\text{Pa}$). This unit can and will be used to quantify stress, not just pressure.

Even though the pascal is a nice Metric unit, it is impractically small. A force of one newton is the force due to gravity acting on just a hundred grams (just a little less than your cellphone, probably). If you can imagine distributing that weight over a square metre then you can imagine how light that feels. One pascal is a small unit for pressure.

In biomechanical and engineering contexts forces of tens or even thousands of newtons are not uncommon. The forces are applied over areas measured in square centimetres or even square millimetres! A smaller area leads to a proportionally larger stress. For this reason we will quite frequently find ourselves using units like

$$1\,\text{GPa} = 10^{+9}\,\text{Pa} = 1\,000\,000\,000\,\text{Pa} \tag{3.3}$$

$$1\,\text{MPa} = 10^{+6}\,\text{Pa} = 1\,000\,000\,\text{Pa} \tag{3.4}$$

$$1\,\text{kPa} = 10^{+3}\,\text{Pa} = 1\,000\,\text{Pa} \tag{3.5}$$

Yes, you are reading those correctly: millions, even *billions* of pascals of stress. The numbers seem large, but that is only because the base unit (the pascal) was so small to begin with.

(As a parallel, imagine that our base unit of distance was equal to the millimetre; what would the distance between Montréal and Toronto look like in those units?)

You probably remember that the atmospheric pressure at sea-level is 101 kPa. The next three examples show how similarly large stresses arise in every-day situations.

Example 3.1 : Standing on my own two feet

I have a mass of about 90 kg. The stress at my ankles is more than the stress at the bottom of my feet. The bottom of my foot measures 30 cm by 12 cm. The shape is rather complicated, but we can estimate its area as a rectangle. My ankles are (approximately) circles of diameter 9 cm. What is the stress at the bottom of my feet? Ignoring the portion of my weight due to my feet, what is the stress at my ankles?

Stress measures how force is distributed across an area. In the case of my feet, the force is my weight and the area is the bottom of *both* feet. So we model the area as *two* rectangles, each measuring 30 cm by 12 cm. If we are going to obtain a stress measured in units of pascals then our areas must be in square metres. So my weight is distributed over an area

$$A = 2 \times (0.30\,\mathrm{m} \times 0.12\,\mathrm{m}) = 0.072\,\mathrm{m}^2 \tag{3.6}$$

at the bottom of my feet. The force is my weight: $F_G = 90\,\mathrm{kg} \times 9.81\,\mathrm{N/kg} = 882.9\,\mathrm{N}$. The stress at the bottom of my feet is thus

$$\sigma = F/A = \frac{882.9\,\mathrm{N}}{0.072\,\mathrm{m}^2} = 12\,262.5\,\mathrm{Pa} = 12\,\mathrm{kPa} \tag{3.7}$$

In the case of the case of my ankles the force is the same (since we neglect the difference between my entire weight and my weight less my feet), but the area is now both ankles. We are told they are approximately circular, each of diameter 9 cm. The area of a circle we know to be given by the formula πr^2.

CAUTION!: The formula for circle area is in terms of the radius, but we are given the diameter. In this example the diameter is 9 cm, the radius to use is 4.5 cm. So the area of one circle is $\pi(0.045\,\mathrm{m})^2 = 0.006\,361\,725\ldots\,\mathrm{m}^2$. The total area of support at my ankles is twice that. Consequently the stress is

$$\sigma = F/A = \frac{882.9\,\mathrm{N}}{2 \times 0.006\,361\,725\ldots\,\mathrm{m}^2} = 69\,391.6\ldots\,\mathrm{Pa} = 69\,\mathrm{kPa} \tag{3.8}$$

The stress at my ankles (69 kPa) is almost six times greater than at my feet (12 kPa) because the area is so much smaller.

When calculating stress (measured in pascals) the area must be in square-metres. If you always include the units of each factor in your calculations they will guide you to perform conversions when they are necessary.

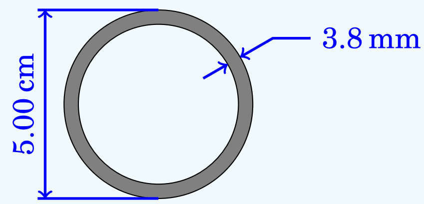
You should re-do that example using your own measurements. Measure the bottom of your shoes, and the diameter of your ankles, to find the appropriate areas.

Example 3.2 : Tubular table leg

A table is supported by four identical legs, each a metal tube. Each tube is circular in cross-section. Each has outside diameter 5.00 cm, and is made from metal that is 3.8 mm thick. If the table (excluding the legs) has weight 29.4 N, what is the stress in the material of each leg?

PICTURE: [[table on four legs]]

When a force is applied to a hollow object the stress is distributed over the place where there *is* material, and not at all in the hollow. The cross-section of one of the tubes looks like this:



The area to use in calculating the stress is the portion colored gray in the diagram. We find the area of the material by subtracting the hollow portion from the entire tube. What we must be careful with is finding the correct *radius* to use for each contribution.

The radius is half the diameter. So the radius of the whole tube is $r_{\text{outer}} = \frac{1}{2} \times 5.00\,\text{cm} = 2.50\,\text{cm} = 0.0250\,\text{m}$. The inner radius of the hollow in the middle of the tube is 3.8 mm less than the outer radius due to the thickness of the material. So the inner radius of the tube is $r_{\text{inner}} = 2.50\,\text{cm} - 3.8\,\text{mm} = 0.0250\,\text{m} - 0.0038\,\text{m} = 0.0212\,\text{m}$. The area of the material is thus

$$\pi r_{\text{outer}}^2 - \pi r_{\text{inner}}^2 = \pi(0.0250\,\text{m})^2 - \pi(0.0212\,\text{m})^2 \tag{3.9}$$

$$= 0.001963495\,\text{m}^2 - 0.001411957\,\text{m}^2 \tag{3.10}$$

$$= 0.000551538\,\text{m}^2 = 5.51538 \times 10^{-4}\,\text{m}^2 \tag{3.11}$$

The weight of the table is distributed equally on the legs. Thus each leg supports a quarter of the weight. The force each leg supports is $\frac{1}{4} \times 29.4\,\text{N} = 7.35\,\text{N}$. Consequently the stress in the material of each leg is

$$\sigma = \frac{F}{A} = \frac{7.35\,\text{N}}{5.51538 \times 10^{-4}\,\text{m}^2} \tag{3.12}$$

$$= 1.332637 \times 10^{+4}\,\text{N/m}^2 = 13.3\,\text{kPa} \tag{3.13}$$

It is important to remember that the rules of significant figures should only be applied to the final results of a calculation, and not to the intermediate steps. In cases where differences are being taken, and the difference is only a small fraction of either number, rounding prematurely can sometimes lead to *very* poor accuracy in the final result.

IMPORTANT : radius versus diameter

When dealing with circular cross-sections or areas be very careful to get the ***radius***. This is important because, typically, we are given the *diameter*. Even if the circular area is exposed (like the *end* of a cylinder) it is usually not possible to find the center of the circle with any accuracy, and measuring the radius is not practical. Thus, when the size of a circular object is measured you usually do so by measuring across its width, and hence get the diameter. Any equations for the circle are expressed in terms of the radius. Be certain to determine the radius from the given information before proceeding.

Example 3.3 : Hanging a mass with an elastic

A 100 gram mass is hanging from an elastic band, as shown in the photograph. The elastic band is 6.2 mm wide and 0.8 mm thick. Measured in *mega*pascals, what is the stress in the material of the elastic?

The mass has a weight of $0.100\,\text{kg} \times 9.81\,\frac{\text{N}}{\text{kg}} = 0.981\,\text{N}$. The cross-sectional area of the band is $6.2\,\text{mm} \times 0.8\,\text{mm} = (6.2 \times 10^{-3}\,\text{m}) \times (0.8 \times 10^{-3}\,\text{m}) = 4.96 \times 10^{-6}\,\text{m}^2$. Looking at the photograph we see that the elastic is a *loop*, and that there are *two* segments supporting the mass. Thus its weight is being supported by the elastic with a cross-sectional area of $2 \times (4.96 \times 10^{-6}\,\text{m}^2) = 9.92 \times 10^{-6}\,\text{m}^2$.

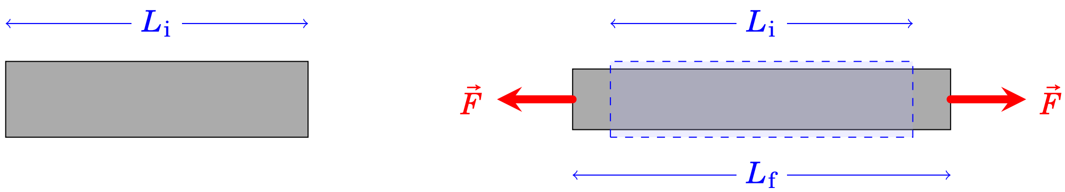
The stress in the material is thus $\sigma = F/A = (0.981\,\text{N})/(9.92 \times 10^{-6}\,\text{m}^2) = 9.89 \times 10^{+4}\,\text{Pa} = 0.00989 \times 10^{+6}\,\text{Pa} = 0.0989\,\text{MPa}$. Noting that we only had *one significant figure* for the width of the band, this result should be written 0.1 MPa.

3.1.2 Deformation

There are many, many different ways that forces can be applied to an object. We will focus on studying only those situations in which the forces applied sum to zero. This is a choice to keep our studies in the context established in chapters 1 and 2 where the forces and torques applied keep the object in static equilibrium. The forces and torques sum to zero, and the object remains in equilibrium, but now we will study how these applied forces and torques cause the shape of the object to change.

Tension

If we grab opposite ends of an object and pull, we say that the object is under ***tension***. If the object is flexible enough, we might see its length *increase* as we stretch it.

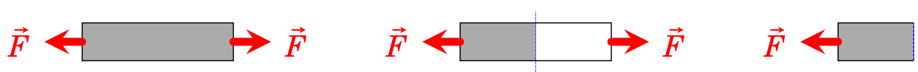


The stress generated by this pair of applied forces is defined to be $\sigma = F/A$. Note that even though there are two forces (one on each end) there is no "factor of two" in the definition $\sigma = F/A$ because the forces actually add to zero. If it helps you understand why there is no factor two, you can imagine that $\sigma = F/A$ is measuring the stress applied across the surface at *one* end of the object, and the other force is just there to keep the object in equilibrium.

Usually, when speaking exclusively about tension, the forces applied at each end are on a common axis. When this is true the pair of forces not only sum to zero, but also apply no net torque to the object. Sometimes, to be explicit about this condition, tension is characterized as "axial tension" (the forces are on a common axis). More commonly this situation is referred to as the object being subjected to *tensile stress*.

Uniformity of Stress

Since the object is in static equilibrium each *piece* of it is also in static equilibrium. This lets us determine the forces and stresses *inside* the object. For example, let's consider the left half of an object under tension:



The whole object, with the two externally applied forces subjecting it to tensile stress.

Here we *imagine* a surface that would cut the object in half.

How does this *piece* of the object remain in equilibrium? What forces are being exerted on it by the other pieces of the object?

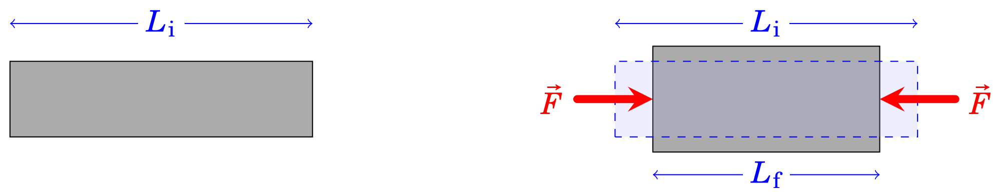
In static equilibrium $\sum \vec{F} = \vec{0}\,\mathrm{N}$ is true for the object as a whole. But if the object as a whole is in static equilibrium, then so too is each piece of it. This means that $\sum \vec{F} = \vec{0}\,\mathrm{N}$ is true for each piece of the object. This means that in the last diagram above there must be a force (or forces) acting at the surface where this piece connects to the rest of the object. This force, an *internal force*, is part of the interactions between the pieces of the object that hold the object together and give it its shape and solidity.

In the diagram below, which pictures the left half of the object, the internal force must balance the externally applied forces. This means that, at *any* cross-section of the object, the internal force must have the same magnitude as the externally applied force. (In the context of strings, ropes and cables, this internal force is what we previously referred to as the *tension*.)

This leads to a very important set of results: If the cross-sectional area of an object is constant along its length, then the stress is the same at each cross-section. But if the cross-section varies along the object's length, then the stress will vary along its length. With the force constant along the length, the stress will larger where the cross-section is narrower, and the stress will be smaller where the cross-section is wider. This is very important in the biomechanical context since bones do not have a constant cross-section! (Consider the shape of the femur as an example.)

Compression

If we grab opposite ends of an object and push, we say that the object is under ***compression***. (Sometimes this situation is referred to as the object being subjected to *compressive stress*.) If the object is flexible enough, we might see its length *decrease* as we squeeze it.

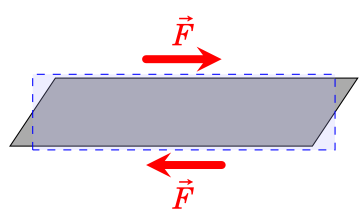


As in the case with tension, the force of compression is the same throughout, but the stress is uniform only if the object has the same cross-section at each point along its length. The *compressive stress* will be greatest where the cross-section is narrowest.

In contrast to tension compression leads to a decrease in length. The other significant difference between tension and compression is the problem of instability. In the case of tension, if the two applied pulls are *not* along the same axis, then they will be applying a *torque* that will turn the object until the forces *are* along the same axis. In contrast in the case of compression, if the two applied pushes are not along the same axis, then they will be applying a torque that will turn the object even *further* from alignment.

Shear

If we place our hands on opposite sides of an object and then move our hands *across* each other (in the way that we might rub our hands together), we say that the object is under ***shear***. If the object is rectangular and is flexible enough, we might see its shape become like a parallelogram.

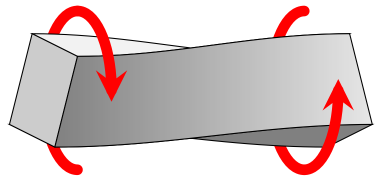


Unlike tension and compression, where the applied forces are perpendicular to the cross-section of the object, shear forces are parallel to the cross-section of the object. The quantity that measures the stress generated by an applied shear still has the form $\sigma = F/A$, but the area A is now the section parallel to the surface where the forces are applied. Imagine spreading out a stack of playing cards, with each card sliding over its neighbors, and you can see the cross-sectional area A is the area of each card. As with tensile and compressive stresses shear is uniform across an object of uniform cross-section.

Note carefully that *pure* shear, as diagrammed above with no other forces, applies a net torque. In reality if an object is subjected to a shear and it remains in equilibrium, other forces must be present to cancel the torque produced by the forces that are producing the shear. In a related circumstance, if tension or compression is applied to an object and the forces are *not axial*, then shear will also be present. All of this is to say that *pure shear* is an idealization. Situations in which shear is applied, and the material is subjected to *shear stress*, do happen, but other forces will be present if the object remains in equilibrium.

Torsion

Pushing and pulling on an object can stretch, compress or shear an object. But how do we *twist* an object? There are two ways.

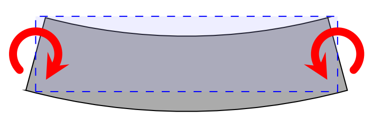


The first (perhaps more natural-feeling?) way is to grab both ends and twist, like we might when wringing water out of a small cloth. Here we can see the object twisting around its length, and its edges now follow a helix. Each hand applies a torque of equal magnitude, about a *common axis*, but of opposite sense. The net torque is zero. The relation between the torques and the object that define this type of deformation is that the axis of rotation that each torque would generate is parallel to the axis of the object.

When torques are applied to an object in this way each segment of the object is trying to rotate opposite its neighbors. For this reason when torsion is applied to an object there is non-uniform shear present. The pieces at the edges are trying to slide the furthest, while the pieces on the axis not at all. The shear is the greatest at the outer surface of the object and zero at its center.

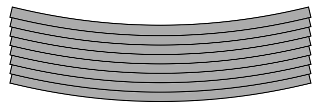
Bending

The second way to twist an object is to grab each end, with your palms facing upwards, and then turn your wrists towards each other. In this case the axis of rotation that each torque would generate is perpendicular to the axis of the object. The sum of torques is also zero in this case since each hand applies a torque or equal magnitude, but of opposite sense. The main difference between this case and previous (where the object was twisted around in a helix) is that the axis of the object ***bends*** away from its equilibrium.

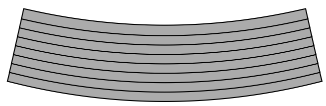


When an object is being bent there is tension on the edge that is being stretched and compression on the edge that is being squeezed. We can see why this is true if we imagine the object being made of layers. If each of the layers were not attached to each other, then they could slide past each other as the object were being bent and each layer could remain the same length. In the actual object each of the layers is attached to its neighbors, and they cannot slide past each other. In the actual object we can see that the layers will have different lengths: those that are longer than before will be in tension and those that are shorter than before will be in compression. This behavior controls how an object that is being bent will break (sub-section 3.2.3).

The undeformed object, visualized as if it were built from separated layers.

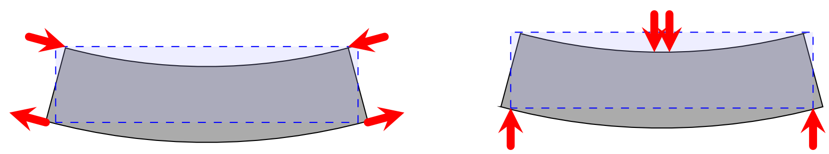


The object deformed by bending stresses. The separated layers each maintain their length. Look closely and notice how the ends are not attached to each other.



If the layers remain attached to each other, then their lengths must become different. The layers on the concave side must become shorter and the layers on the convex side must become longer. Again, look closely at the ends and now notice how they remain attached to each other.

The forces that would manifest this distribution of stresses and consequent deformation: In the next diagrams we see how a bending deformation can be produced by either a pair of applied torques (at each end) or a pair of shear forces applied to each half.

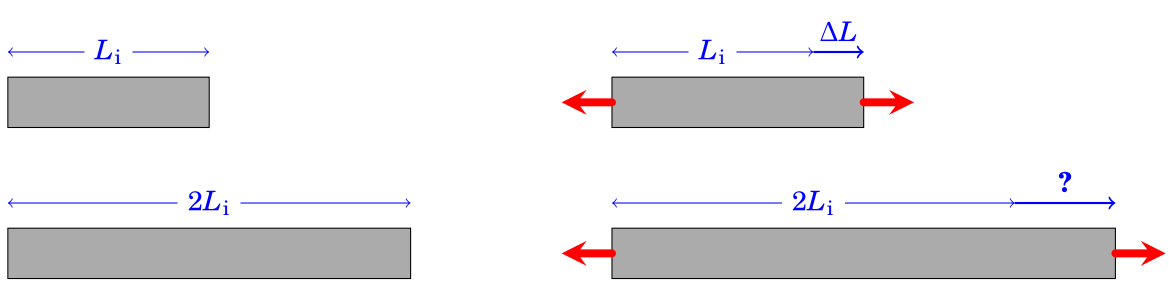


From the details shown above we can see that applied torques will always generate shear stresses in a material. Conversely we've seen that the applied forces that generate shear stresses have an associated torque. Torque and shear are inter-related, but are not identical. Whenever torque is applied to an object, there will be torsional and/or bending deformations. Whenever torque is applied to an object, there may be tensile or compressive stresses generated, but there will always be shear stresses generated in the material.

3.1.3 Strain

With the different *types* of deformation mapped out in the previous sub-section, we now turn to quantifying how we will *measure the amount* of deformation.

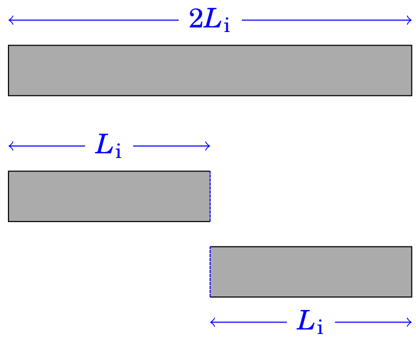
As a specific case, let's consider two objects made from the same material, with the same cross-sectional area, but with one of them being twice as long as the other (they have lengths L_i and $2L_\mathrm{i}$). If we apply the same amount of axial tensile stress, the length of both objects will increase. Let's call the amount of the change in length of the shorter object ΔL. By how much will the (initially) longer object stretch?



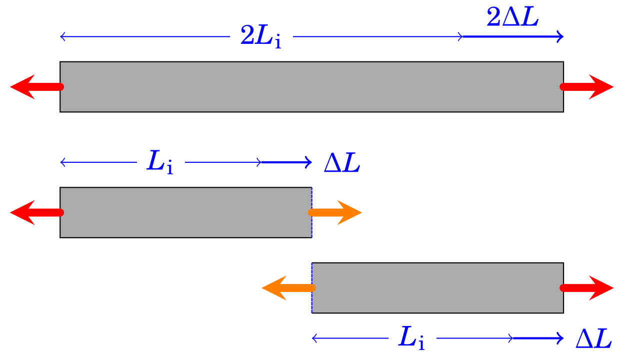
The two objects, initially.

The two objects, with equal stress.

Each *half* of the initially longer object are identical to the initially shorter object. Because of this, when subjected to the same tensile stress, each half will stretch by the same amount (ΔL). Thus, in total, it will stretch twice as much ($2\Delta L$), as shown in the diagram below.



The longer object, initially, visualized as two identical halves.



The red arrows are the *externally applied* forces that are exerting the tensile stress. The orange arrow are the *internal tension* that keep the two halves attached.

We could repeat this argument in cases where the initially longer object was longer by some other factor (like 3, or 7, or 42.1). In all those cases the amount by which it stretches is proportional to the factor by which it was initially longer. That is: *for the same stress the amount of deformation is proportional to its initial length*. This shows that the *ratio* of the change in length to the initial length will be independent of the initial length!

For axially applied forces the ***strain*** is defined to be the change in an object's length relative to its original un-stressed length. (For changes in the shape of an object due to shear or torsion the deformation is characterized by the *angle*, but we will not study those cases quantitatively in this course.) The symbol used to denote strain is the Greek letter epsilon (ϵ), and the mathematical statement that defines strain is written

$$\epsilon = \frac{\Delta L}{L_\mathrm{i}} \tag{3.14}$$

where "L_i" is the *initial*, un-stressed length of the object. The collection of symbols "ΔL" does ***not*** mean "$\Delta \times L$". It means the change in the length, specifically the *difference* between the final and initial lengths: $\Delta L = L_\mathrm{final} - L_\mathrm{initial}$. Because of that the strain of an object is calculated by

$$\epsilon = \frac{L_\mathrm{final} - L_\mathrm{initial}}{L_\mathrm{initial}} \tag{3.15}$$

An example will show how this is used in practice.

Example 3.4 : Rope stretch

A rope of length 3.050 m is pulled from both ends. The rope stretches to a length of 3.062 m. What is the strain of the rope?

The strain is a ratio of lengths, so our answer will have no units. Since the change in length is small in comparison to the original length, we expect the strain to be a small number. Since the length increases, the strain must be positive.

The change in the rope's length is $\Delta L = L_\mathrm{f} - L_\mathrm{i} = 3.062\,\mathrm{m} - 3.050\,\mathrm{m} = +0.012\,\mathrm{m}$. (If we concern ourselves with the number of significant figures we note that, even with the lengths given with four significant figures, the difference has only *two*.) The strain is thus

$$\epsilon = \Delta L/L \tag{3.16}$$

$$= +0.012\,\text{m}/3.050\,\text{m} \tag{3.17}$$

$$= +0.0039 = +3.9 \times 10^{-3} \tag{3.18}$$

The strain is thus $\epsilon = +3.9 \times 10^{-3}$.

There are several things to note about this result:

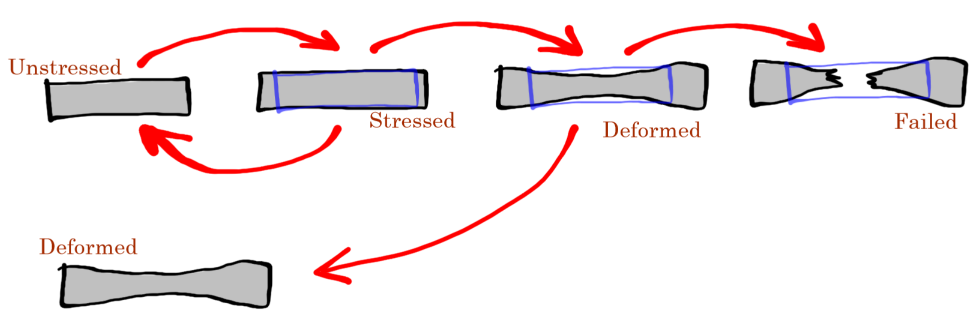
(1) Strain is a length divided by a length, so it has no units; it is just a number. Since it is a ratio of lengths it is critical to express both the numerator and denominator in terms of identical units of length before dividing. (Refer back to example 0.1.2.)

(2) The strain is very small in magnitude. This is typical for solid materials. If you ever calculate a strain that is bigger than one, and the material is not a soft elastic, then you can be sure you've made a mistake – or the object has completely broken! Since the strain is usually a very small number, strains will usually be written using scientific notation which shows the power of ten explicitly; as in this example with the factor 10^{-3}.

(3) The sign tells us if the length has increased or decreased. A positive strain is when the final length is greater than the initial, when the object's length has increased. If the strain were negative, the final length would be less than the initial, which would indicate that the object's length had decreased. As with vector components it is a good habit to explicitly include the sign of the strain, even when it is positive.

3.2 Everything Breaks

Everything is made of atoms, held together by chemical bonds. Each chemical bond has a bonding energy – the amount of effort required to separate the bonded atoms. For this reason every material can be broken by a finite amount of effort; there is no such thing as an "indestructible" material. There is an upper limit to the amount of mechanical energy that the material of an object can absorb before the object becomes permanently deformed or broken.



Below that limit all of the work done to it is returned when the stress is removed. This fact will be explored in more detail in chapter 4 when we develop the topic of *Energy*.

3.2.1 Elastic Regime

Every solid material has a range of stress for which, when that applied stress is removed, the object returns to its original shape, and all of the work done to it is returned when the stress is removed. The *range* of deformation for which the object will return to its original shape is called the ***elastic regime***. This physical characteristic of a material is referred to as its *elasticity*.

Generally, for small amounts of applied stress the amount of deformation (measured by the strain) is proportional. (In your high-school physics you should have see this as *Hooke's Law* for idealized springs, which relates force to extension: $F = kx$.)

3.2.2 Yield & the Plastic Regime

For most materials there is a point beyond which increasing the applied stress *begins* to change the microscopic structure of the material. Molecular bonds are rearranged or broken. In materials where it is possible, pieces shift past each other. This is called the ***yield point***. Past the point of yield, when the applied stress is released, the object does not return to its original shape. This deformation is characterized as being *plastic*. The physical characteristic of a material that quantifies how easy it is to deform in this way is called its *plasticity*.

(Here, because of history, we have two related words: the physical characteristic of plasticity, and the noun plastic. The category of materials commonly referred to as "plastic" – like the polymeric materials used to make water bottles and cellphone cases – have that name because they are easy to deform plastically. There is no deeper meaning.)

The property of plasticity is utterly critical to modern manufacturing since it is what allows us to shape metals and polymeric materials. Cars, containers, computers, even knives and forks, are objects that could not be made if we could not deform the materials.

In the biomechanics, the value or danger of plastic deformation is strongly dependent upon the amount of deformation and the anatomical segment that is being deformed. When exercising a very small amount of plastic deformation is necessary to stimulate growth, and subsequent strengthening. But large amounts of deformation of tissues like tendons and ligaments, like sprains, are not easy to recover from.

3.2.3 Failure

All materials are held together by their chemical and inter-molecular bonds. Every bond of this type represents a finite amount of energy. Consequently ***every*** material can be disrupted and broken by a finite amount of effort. There is no such thing as an indestructible material.

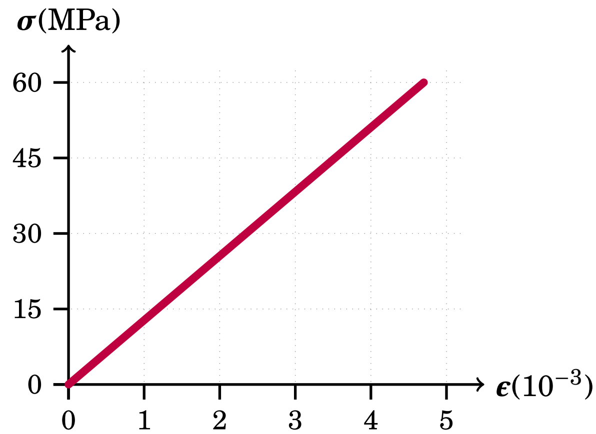
The *way* in which a material fails depends amount the type of material and the type of stress to which it is subjected. There are *almost* no simple rules about what failure will look like – though we will look a little closer at this in section 3.4.

3.3 Stress-Strain Diagrams

The old wisdom is that "A picture is worth a thousand words". The scientific fact is that "A graph can contain thousands of data points". For this reason how things bend and when things break are data that can be expressed graphically. Our goal is to learn how to read a stress-strain diagram to extract such information.

3.3.1 Stress-Strain Relationship

When forces are applied to an object its deformation can be measured. From the object's geometry the corresponding strain and stress can be calculated. As the applied forces are varied and the resulting deformations vary, the accumulated data can be graphed. Conventionally the stress is plotted as a function of the strain.



Example of a stress-strain graph: Plotted is the stress, σ (measured in units of megapascals), as a function of the strain, ϵ (which is dimensionless, but plotted as multiples of $10^{-3} = 0.001$).

It might seem strange to plot the relation this way. It's correct to think that, realistically, we control the force applied (thus the stress), and as a result of that there is a deformation (the resulting strain). So you might think that the stress should be the independent variable on the horizontal axis. However, the *purpose* of the stress-strain diagram is to tell us what the stress (force) is ***in*** the material when we have a measurement of the strain (deformation) of the object. For this reason, stress-strain relationships are plotted with stress as a function of the strain.

Reading a Stress-Strain Diagram

The most important fact to remember about a stress-strain diagram is that it is the graph of the properties of a *material*, not of a specific object with a specific geometry. Remember that force was converted to stress by factoring out the area and the that deformation was converted to strain by factoring out the length (in the case of axial stresses). Stress and strain relate in a manner that depends only on the properties of the material, and that is independent of the object's geometry.

To read any information from a stress-strain diagram you must establish the context. What is the category of applied stress?: tension, compression, shear, torsion, bending? (Recall the categories of deformation listed in 3.1.2.) You must know the context to order correctly interpret the data of the graph. Usually this information is not written on the graph itself, but is found either in the caption to the graph, or in the surrounding text; be certain to find this before trying to extract data from the graph. In this course we will limit our quantitative analyses to axial stresses of either tension or compression. The more complex cases of shear, torsion and bending will be considered qualitatively only.

Read the units carefully. Strain is a number (no units), but will typically be expressed as a multiple of a small number, or sometimes expressed as a percent. Stress is usually measured in very large multiples of the pascal, typically megapascals or gigapascals. Be sure to read the units on the axes of the stress-strain diagram very carefully.

If you must apply the information obtained from a stress-strain diagram to an object with a specific geometry, remember that the deformation can be obtained from the strain ($\Delta L = \epsilon L_{\text{i}}$) and that the force can be obtained from the stress ($F = \sigma A$). In this process of calculation be certain that: the length L_{i} of the object's geometry is along the axis of the applied stress; and, the area A the cross-section perpendicular to the axis.

3.3.2 Stiffness

How much effort is required to change an object's shape? To quantify this we would ask what amount of force is required to achieve a specific amount of deformation. This measurement would quantify the *stiffness* of the material.

The steeper the graph, the stiffer the material. The value of the slope of tangent to the graph at any point is called the ***Young's Modulus*** of the material in that state:

$$\text{Young's Modulus} = \frac{\Delta\text{stress}}{\Delta\text{strain}} \tag{3.19}$$

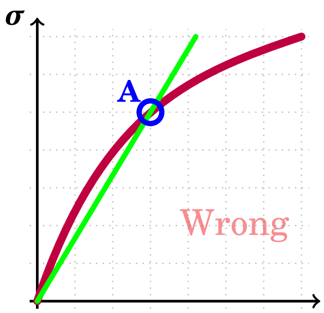
Mathematically this is defined as

$$\mathscr{Y} = \frac{\Delta\sigma}{\Delta\epsilon} \tag{3.20}$$

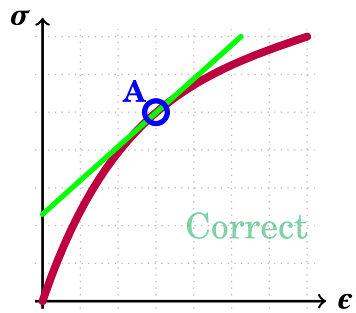
(Note that Young's modulus is also sometimes referred to as the material's *modulus of elasticity*. For that reason you may sometimes see the symbol "E" used for this quantity.) The larger the value of $\mathscr{Y}$ the stiffer an object made of that material will be.

"Stiffness", defined in this way, will not depend upon the geometry of the object that we measure. It does not depend upon the length of the material since that was factored-out by the definition of strain. It does not depend upon the cross-sectional area since that was factored-out by the definition of stress. It will depend only upon the material.

The value of Young's modulus is a property of the material that does not depend upon the geometry, but it does vary with the strain of the material.



The point "A" circled in blue has coordinates (ϵ,σ). The slope of the green line, which starts at the origin, equals the ratio $\Delta\sigma/\Delta\epsilon = \sigma/\epsilon$. This is *not* the value of $\mathscr{Y}$ when the material is in the state "A".



The green line here is *tangent* to the stress-strain curve at the point "A". The slope of the green line *is* the value of $\mathscr{Y}$ when the material is in the state "A". $\mathscr{Y} = \Delta\sigma/\Delta\epsilon$ is calculated using two points chosen on the green tangent line.

The differences denoted by the "Δ"s in the definition (3.20) are the differences between two separated points on the line that is tangent to the stress-strain curve. Note that if the stress-strain relationship is ***not*** a straight line then that slope is not equal to the ratio of the coordinates: $\mathscr{Y} = (\Delta\sigma/\Delta\epsilon) \neq (\sigma/\epsilon)$. This is shown in the diagram above, on the right.

Values

The table below presents some examples of the values of Young's Modulus for some typical materials.

Table 3.1: Young's modulus ($\mathscr{Y}$), examples of ranges of values.

Material	$\mathscr{Y}$ (GPa)
Ceramics	$60 \cdots 1000+$
Porous Ceramics	$8 \cdots 100$
Glass	$50 \cdots 90$
Metals & Alloys	$13 \cdots 400$
Composites	$8 \cdots 200$
Wood	$0.08 \cdots 25$
Polymers	$< 0.01 \cdots 10$
Polymers Foams	$< 0.01 \cdots 0.5$
Rubbers	$< 0.01 \cdots 0.1$

All of these entries have large magnitudes due to the fact that the pascal is a very small unit of pressure. Since stresses are measured in pascals, the numerator of the expression for the slope will typically be very large. This is compounded by the fact that the strain of most materials (below their yield point) are very small, numerically. This makes the ratio very large.

As you might expect, biological structural materials (like tendons, ligaments, and bones) are *extremely* variable in their composition and properties. Their properties vary slightly from person to person, but vary greatly depending upon which portion of the anatomy they are sampled from, and even along which axis they are stressed! Consequently it is difficult to meaningfully assign a single value of elastic modulus to classes of material like "bone". For our purposes it is enough to know that most biological structural materials have values that fall in the category of ***composites***, which is in bold in the table above.

Hooke's Law

In high school science you would have studied springs and elastics. The formula that describes the relation between the deformation (x) of the spring and the force (F) the spring exerts on the thing causing the deformation is: $F = kx$. This formula is known as *Hooke's Law*.

As explained in sub-section 3.2.1, for sufficiently small stresses the object will return to its original shape when the applied stresses are removed. For almost all solid materials the stress and strain are *linearly* proportional to each other in this regime of small deformations. When this is true the relation between the applied force (F) and the resulting deformation (ΔL) is

$$\mathscr{Y} = \frac{\Delta\sigma}{\Delta\epsilon} = \frac{\sigma - 0\,\mathrm{Pa}}{\epsilon - 0} = \frac{\sigma}{\epsilon} = \frac{F/A}{\Delta L/L} = \frac{L \times F}{A \times \Delta L} \tag{3.21}$$

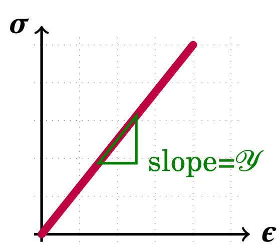
Solving this to isolate the force, we get

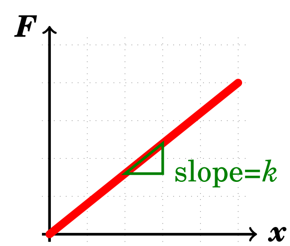
$$F = \left(\mathscr{Y} \times \frac{A}{L}\right) \times \Delta L \tag{3.22}$$

Comparing this with Hooke's Law ($F = kx$), since $x = \Delta L$ we can see that the spring constant is

$$k = \mathscr{Y} A/L \tag{3.23}$$

This is how the stiffness of the spring is determined by the object's material and its geometry. In a sense, this derivation takes us backwards through the process we used to define $\mathscr{Y}$.

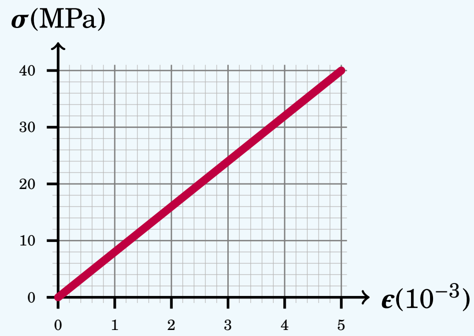




We began by asking how to speak objectively of the "strength" of a material. We knew that applying a force changed the shape of an object, but that the relation between those quantities depended not just on the material, but on its shape. So, to answer the question, we found a way to remove the factors of the object's geometry. By factoring out the cross-sectional area from the force we obtained the stress ($\sigma = F/A$), and by factoring out the length from the deformation we obtained the strain ($\epsilon = \Delta L/L$). The relation between stress and strain was encoded in Young's modulus $\mathscr{Y}$.

But if we want to return to the specific instance of applying forces to an object of that material with a specific geometry, the relation between force and deformation is Hooke's Law. When we specify the geometry we can map stress onto force ($F = A\sigma$) and strain onto deformation ($x = L\epsilon$). The preceding result then shows the inter-relationship between the value of Young's modulus (which is a property of the material) and Hooke's Law (which is a property of an object made from that material, with a specific shape): $k = \mathscr{Y} A/L$.

Example 3.5 : Building material



Wood is an inexpensive building material that is easy to shape and join. The graph on the left shows the elastic regime of its stress-strain relation for when the wood is under *compression*. In the graph stress σ is measured in units of megapascals, and the strain ϵ (which is dimensionless) is plotted as multiples of $10^{-3} = 0.001$.

A standard shape of wood used in house construction is the "two-by-four". This is a piece of wood that measures $2\,\text{in} \times 4\,\text{in}$ in cross-section, and has a length of eight feet ($8\,\text{ft} = 8 \times 12\,\text{in}$). If this piece of wood is supporting 11 785 kg:

- **(a)** What is the stress is the wood, measured in megapascals?
- **(b)** From the graph, what is the Young's modulus of this material?
- **(c)** From the equation for the line, what is the strain in the wood?
- **(d)** What is the change in length of the wood, measured in inches and in millimetres?

Notice that we have been given our information in a mixture of imperial units and Metric; we will have to be careful with the units!

Reading the description of the graph carefully we note that the material is under compression. (This graph would be in the *third quadrant* of the stress-strain plane, but has been transposed up into the first quadrant for ease of use.) This means that the answer for the change in the length will be negative. When we read a value from the graph we will have to remember that both the ϵ and σ are, in fact, negative when considering compression. So reading "$\sigma = 37\,\text{MPa}$" on the graph axis would actually mean $\sigma = -37\,\text{MPa}$.

Part(a) To find the stress in megapascals, we need the force in newtons, and the area in square-metres. The wood is supporting a mass of 11 785 kg, which corresponds to a weight of $11\,785\,\text{kg} \times 9.81\,\text{N/kg} = 0.115\,61 \times 10^{+6}\,\text{N}$. The cross-section of the wood measures two inches ($2\,\text{in} \times 0.0254\,\text{m/in} = 0.0508\,\text{m}$) by four inches ($4\,\text{in} \times 0.0254\,\text{m/in} = 0.1016\,\text{m}$), which is an area of $0.0508\,\text{m} \times 0.1016\,\text{m} = 5.161\,28 \times 10^{-3}\,\text{m}^2$. Remembering that this is under compression, the stress is thus

$$\sigma = -\frac{F}{A} = -\frac{0.115\,61 \times 10^{+6}\,\text{N}}{5.161\,28 \times 10^{-3}\,\text{m}^2} = -22.400 \times 10^{+6}\,\text{Pa} = -22.400\,\text{MPa}$$

Part(b) The relationship in the graph between stress and strain in the material is a straight line through the origin ($\epsilon = 0$ and $\sigma = 0\,\text{MPa}$). This means that the Young's modulus for the material (which is the slope of this graph) can be calculated by choosing the origin and the point at the far end of the line. The point at the top-right end of the line has coordinates $\epsilon = -5 \times 10^{-3}$ and $\sigma = -40\,\text{MPa}$ (where we have remembered that the material is under compression). The Young's modulus is thus:

$$\mathscr{Y} = \frac{\Delta\sigma}{\Delta\epsilon} = \frac{(-40\,\text{MPa}) - (0\,\text{MPa})}{(-5\times10^{-3}) - (0)} = +8\,\text{GPa}$$

Part(c) Looking at the graph, we see that a stress of $-22\,\text{MPa}$ corresponds to a strain somewhere between -2.6×10^{-3} and -2.8×10^{-3}. (This is not accurate, but we read from the graph to obtain an estimate of what our calculated answer will have to be.)

Since the straight line of the stress-strain relationship passes through the origin, for any point on this portion of the curve we can write

$$\mathscr{Y} = \frac{\Delta\sigma}{\Delta\epsilon} = \frac{\sigma - 0\,\text{MPa}}{\epsilon - 0} = \frac{\sigma}{\epsilon}$$
$$\epsilon = \frac{\sigma}{\mathscr{Y}} = \frac{-22.400\,\text{MPa}}{+8\,\text{GPa}} = \frac{-22.400\times10^{+6}\,\text{Pa}}{+8\times10^{+9}\,\text{Pa}} = -2.8\times10^{-3}$$

This is within our estimate, so we are confident of the result.

Part(d) Given the initial length of the wood ($L_{\text{i}} = 8\times12\,\text{in} = 96\,\text{in}$) and the strain, the change in the length is

$$\Delta L = \epsilon L_{\text{i}} = (-2.8\times10^{-3})(96\,\text{in}) = -0.27\,\text{in}$$

In millimetres this is $\Delta L = (-0.27\,\text{in})(25.4\,\text{mm}) = -6.8\,\text{mm}$.

3.3.3 Yield & Deformation

The stress (or strain) at which a material yields can be considered a second way of measuring its strength.

Table 3.2: Yield stress, examples of ranges of values.

Material	*Yield stress* (MPa)
Ceramics	80 ⋯ 10 000 <
Porous Ceramics	10 ⋯ 1100
Glass	212 ⋯ 440
Metals & Alloys	4 ⋯ 3000
Composites	50 ⋯ 1600
Wood	0.4 ⋯ 80
Polymers	5 ⋯ 80
Polymers Foams	< 0.1 ⋯ 10
Rubbers	2 ⋯ 30

In the chapter on Energy we will return to this table to understand why *ceramics* are not useful as a building material despite their large values of yield stress.

3.3.4 Failure

The stress (or strain) at which a material fails (breaking into two or more separated segments) can be considered a third way of measuring its strength.

Table 3.3: Strain at Failure, examples of ranges of values.

Material	*Strain at Failure* (#)
Ceramics	$0.0003 \cdots 0.01$
Porous Ceramics	$< 0.0001 \cdots 0.0016$
Glass	$0.0004 \cdots 0.0005$
Metals & Alloys	$0.0005 \cdots 0.9$
Composites	$0.008 \cdots 0.05$
Wood	$0.003 \cdots 0.05$
Polymers	$0.008 \cdots 5$
Polymers Foams	$0.002 \cdots 10$
Rubbers	$1 \cdots 9$

It is interesting to note that the order of this table, from greatest to least, is almost opposite that of table 3.1. This means that (typically) the materials that are easiest to deform are the ones that deform the most before breaking, and that the stiffest ones are the ones the deform the least before breaking. Hopefully this will seem like common-sense to you.

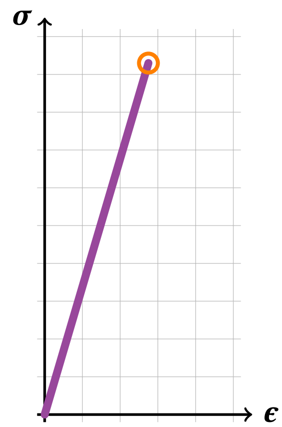
The exception to that trend are metals, which have high $\mathscr{Y}$ (but less than ceramics) while having strains at failure greater than some composites! It is for this reason that metals are used engineering.

3.4 Some Categories of Materials

[[Table: values of Young's Modulus, Yield stress, and Failure for some typical materials]] Comments comparing the ultimate stress, stress at failure and the typically large gap between those values and the Young's Modulus.

3.4.1 Brittle Materials

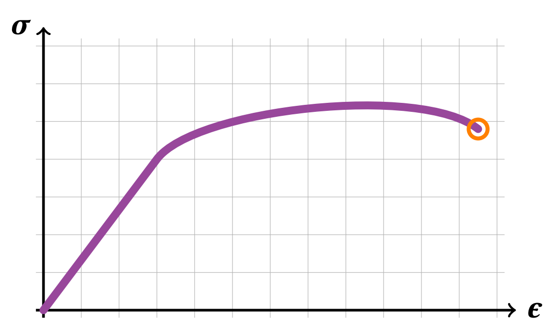
Glass and other covalently bonded materials, like ceramics and crystals, are usually very stiff. Their elastic deformation is usually so small as to be practically invisible to human eyes. Aside from being very stiff their one defining characteristic is that their yield point is also the point of failure, and that that failure is usually catastrophic. A material with this behavior is referred to as ***brittle***. While withstanding a large stress a brittle material might appear the same as its unstressed state. But a slight increase in the applied stress past its point of failure would lead to the material suddenly breaking, across its entire volume, into separate pieces (into at least two, but sometimes *thousands*).



Concentration of Stress

In brittle materials when failure begins it manifests as a *crack* at some location in the material. In the volume surrounding that crack the stress is *larger* than just previously because the crack is a surface where the material is no longer connected to itself. This is a reduction in the cross-section of material across which the force in the material is distrbuted, and thus an increase in stress occurs. With the material at the beginning of failure, this increase in stress puts it even further into the failure regime, and the crack becomes even larger, propagating across the material until it breaks into two (or more) pieces.

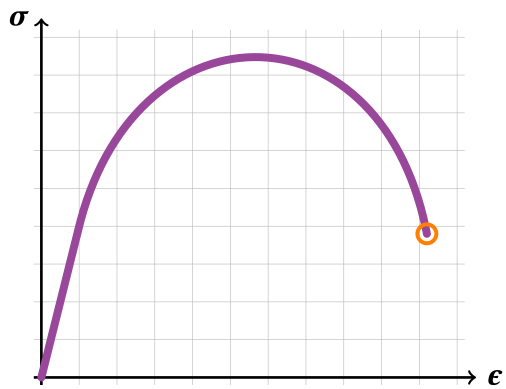
3.4.2 Plastics



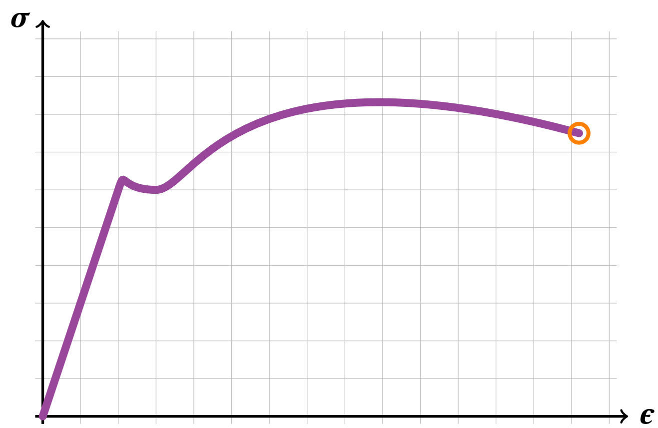
3.4.3 Metals

[[diagram: Stress-Strain, initially linear, followed yield, followed by ductile region]] [[diagram: Stress-Strain, initially linear, followed yield, followed by ductile region, followed by strain-hardening regime]]

Ductile metals like copper.



Metal alloys that exhibit strain-hardening, like steel.

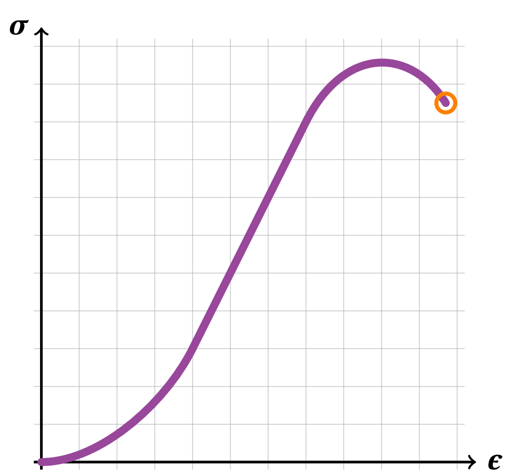


The exception of brittle metals, like chemically pure iron.

3.4.4 Elastomers

Rubber, tendons, collagen.

[[diagram: gradual straightening of crumpled elastic, followed by elastic stretching]]



The “toe” of the stress-strain curve and the aligning of the elastic fibers.

3.4.5 Composite Materials

"It's complicated."

Bone

Asymmetric stress-strain curve: the failure modes in the tensile stress regime are different from the failure modes in the compressive stress regime.

You'll see this in your Biomechanics class. Come back here to review the tools you'll need to read those diagrams.

Concrete

Large compressive strength, very little tensile strength.

Foams

Styrofoam. Foam metals (Eg.Porous tantalum.) used in bone implants.

The curves for materials of this type under compression begin with a regular-looking elastic regime. But most of the material's volume is bubbles. The walls of material that surround those bubbles are thin, and thus provide very little support against applied stress. Consequently the yield point happens much sooner for the foam that it would for a solid piece of the same material.

Once beyond the yield point, the bubbles in the foam begin to collapse under the applied stress. This appears as a plateau on the stress-strain curve: at an almost constant stress the bubble progressively collapse.

Approaching the stage where almost all bubbles have been collapsed the properties of the solid material re-emerge. This leads to a significant increase in the object's stiffness.

Chapter 4

Energy

I am going to help you learn what energy is.

But I have a problem: Energy is not a *thing*.

Energy is not a solid, like a block of wood that you can hold or throw. Energy is not a liquid, like water that you can pour or dip into. Energy is not a gas, like air you can blow into a balloon and then squeeze. If I am going to teach you about energy, I can't pass some to you and say "here, this is energy". You can't touch energy.

So I am lucky that touch is not your only sense.

You can feel that a fire is hotter than your body, and that your body is hotter than ice. To throw a ball faster, you know that you would have to throw it harder. You can feel that it takes more effort to move furniture across a room than to slide a book across a table. If something falls onto your foot, you know that the further it falls, the more it will hurt. You know that it is easier to whisper than to shout, and that it is easier to hear the shout than the whisper. You can see that your computer screen is not as bright as the Sun.

Energy is as real as a solid object, a liquid substance or a gas – but energy is neither a solid, a liquid nor a gas. Energy is something else. But it is real, and so can be precisely measured.

Your sense of how hot or cold a thing is relates to the measurable quantity temperature, and temperature is an indicator of the thermal energy in an object. The effort you make to throw an object faster changes its measurable speed, which relates to its kinetic energy. The effort that you make to slide a heavy piece of furniture relates to the strength of the friction between the furniture and the floor, which dissipates your effort as thermal energy into the surfaces of contact, vibrations in the floor, and sound in the air. The amount of pain or damage you experience when something falls on your foot relates to the amount of gravitational energy that has been transformed into kinetic energy that is then dissipated into your flesh and bones.

This list of relations between our perceptions and the objectively different *forms of energy* is meant to connect those things you know to the ideas that we will be studying.

4.1 Categories & Forms of Energy

The different forms of energy fall into two categories: Motion and Interaction.

4.1.1 Motion

The energy associated with motion is referred to as *kinetic energy*. In a real sense this energy is the "effort" we put into making an object move. You may recall this type from high school: $K = \frac{1}{2}mv^2$. That expression applies when the object (of mass m) is moving, *as a whole*, with a speed v. Each piece is moving in the same direction and at the same speed – moving with the same *velocity*, $\vec{v}$.

Coherent

But there are other forms of kinetic energy which are possible if the pieces of the object are moving differently in relation to each other. It is possible for the different parts of an object to be moving at different speeds and different directions, but in a way that is synchronized. When the speeds and directions of motion of the pieces relate to each other ***coherently***, when the interrelationships between their velocities $\vec{v}$ do not change with time, motions like rotation and oscillation can manifest.

Incoherent

At the other extreme, when the speeds and directions of motion of each of the pieces of the object are unrelated to each other, the motion is random or ***incoherent***. When this randomness of motion is at the molecular level this form of kinetic energy is referred to as *thermal energy*. This form is "referred to" as thermal energy because it ***is*** thermal energy. In later sections we will study the relationship between this energy and *temperature*.

Units of Energy

While the velocity $\vec{v}$ of an object is vector, energy (which depends only upon the magnitude of the velocity) is just a number. From the expression for kinetic energy $K = \frac{1}{2}mv^2$ we can see that the units of energy are mass times the square of speed. This combination is called the ***joule*** (symbolized as J), and it is the SI unit of energy.

Conventionally it is defined in terms of mechanical interactions: the energy transferred by a force of one newton acting through a distance of one metre is one joule

$$1\,\mathrm{J} = 1\,\mathrm{N} \times 1\,\mathrm{m} = 1\,\mathrm{kg} \cdot \mathrm{m}^2/\mathrm{s}^2 \tag{4.1}$$

Despite being defined relative to mechanical work all forms of energy (thermal, gravitational, etc) can be transformed into one another through interaction, and so are all measured relative to the same unit.

4.1.2 Interaction

Back when we first discussed the types of forces (1.2) we encountered the two categories of forces: contact, and non-contact. Here we discuss the underlying *fundamental interactions* that are the cause of those forces.

Gravity

Objects with mass attract each other. This interaction is known as ***gravity***. This interaction is so weak that it can not be noticed unless at least one of the objects has the mass of a planet. There are instruments that can measure these very small forces (like that between a small piece of metal and a mountain), but on the human scale only the interaction with the Earth as a whole matters.

Electromagnetic

Objects with charge attract if they are opposites and repel if they are the same. These forces are referred to as *electric* forces. Charges that are *moving* exert *magnetic* forces on each other. The magnetic interaction is usually treated mathematically as if it were separate from electric interactions, but it is, in fact, physically an actual part of electric interactions. For this reason they are referred to by a single name: ***electromagnetic*** interactions.

Electromagnetic forces are stronger than gravity by an *enormous* factor. Because charge is balanced in atoms with equal numbers of protons (positively charged) and electrons (negatively charged), we almost never see the direct strength of electric forces. When charges *do* become separated, the enormous force between them causes them to rejoin almost immediately. One of the controlled exceptions to that rule is the subject of chapter 6, where we will study electric circuits.

Nuclear

The profoundly small and dense center of an atom is called the atomic nucleus. The atomic nucleus is a collection of protons (positively charged) and neutrons (zero charge). The electric repulsion between all the protons would explode the nucleus, except that there is a profoundly strong attractive interaction between all the pieces of the nucleus. Changes of electron interactions between atoms in molecules corresponds to changes in chemical energy, and in reactions like combustion can release heat and light. Similarly, changes in nuclear interactions can release heat and light. But, because nuclear interactions are so much stronger, the amount of energy released is proportionally much larger, as in nuclear explosions, or in the burning of the Sun. ***Nuclear*** interactions are not directly relevant in your studies towards becoming a Physiotherapy Technologist, and are mentioned here only for the sake of completeness.

Contact

It is a fact that two objects can not occupy the same space. This fact manifests as a repulsion between all pieces of matter that acts only at atomic distances. It is this interaction that allows force and energy to be transferred through “pushes” and “pulls" between objects in contact. “***Contact***”, described this way, is similar to the other interactions in that it can transfer or transform energies. But it differs from interactions like gravity and electromagnetism in that it can not *store* energy when objects are separated since it does not exist in the space between objects, but only where they touch. When objects (specifically molecules, atoms, or other fundamental particles) are brought into contact, energy can be stored by this interaction as a contribution to the *pressure* (see section 4.6).

Transfer and Transform

Energy can be transferred and/or transformed by interactions. The easiest example of this is how a contact interaction can transfer kinetic energy from one object to another in a collision. A fundamental example is how friction transforms the kinetic energy of a sliding object into sound and thermal energy. A common example is when something drops, converting gravitational energy into kinetic energy. A more complex example might be how electromagnetic energy can be transformed through magnetic repulsion into the kinetic energy of a projectile, which also increases the gravitational energy between the projectile and the Earth as it flies upwards.

At the common surface of contact between two objects the normal force is generated. This is the interaction that allows us to change the speed, and hence kinetic energy, of an object. This contact interaction is what transfers energy when we push on an object.

If two objects are sliding across each other, then there is friction. This contact interaction transforms the coherent kinetic energy of the objects' motion (each as a whole) into the incoherent kinetic energy of their constituent molecules. This contact interaction transforms macroscopic motion into thermal energy.

Energy is stored in the interaction.

Power

▲**FIX:** HERE: Move the contents of subsection 4.3.4 to here.

4.1.3 Forms of Energy

The different forms of energy fall into the two categories of motion and interaction. The category of "motion" encompasses all the forms of kinetic energy that are associated with all the different modes of motion. The category of "interaction" encompasses all the forms if energy associated with the four basic forces of gravity, electromagnetic, nuclear, and contact. All other *apparent* forms of energy are, in fact, combinations of these fundamental forms.

Motion	**Interaction**
Coherent: • Linear Kinetic • Rotational Kinetic • Oscillation / Vibration Incoherent: • Thermal	Gravity Electromagnetic Nuclear Contact

Sound is not a separate form of energy. Sound is actually a variation in the kinetic energies and density (hence pressure) of the molecules in the air. The way in which this energy is transported from one place to another by ***waves*** will be the focus of our study in chapter 5.

Light that we can see and the heat radiated by very hot objects are not separate forms of energy. They are both instances of energy being transferred by electromagnetic interaction; specifically by *electromagnetic waves*.

Chemical energy is not a separate form of energy. The electrons and atoms in molecules have kinetic energies. The electrons and atoms in molecules also have energies due to their electric and contact interactions. A molecule can spin, and its shape can vibrate, which are forms of kinetic energy. Chemical energy is the *portion* of all these combined energies that can be changed when the arrangements of the electrons between atoms changes – when there is a chemical reaction.

Related to chemical energy is the way in which energy is required to deform the shape of atomic orbitals and molecular bonds. When the stress in a material is zero, the atoms in the molecules of the material are a specific distance apart: their *equilibrium distance*. To change that distance – to either try separating the bonded atoms, or to try pushing atoms into each other – requires work. That work is stored in the deformed bonds as *elastic* energy.

4.1.4 Conservation of Energy

When we keep track of all the different forms of energy present we find, experimentally, that their total does not change with time. This fact is referred to as the Principle of the Conservation of Energy. Of all the physical laws, this is one of the most important because of how it applies universally. Of all the physical laws, this is one of the most useful because of its mathematical simplicity.

Specify what constitutes the System. Identify what objects and entities are inside (are a part of) the system, and what are outside.

Identify what forms of energy are present in the system. Then name the interactions present that can transfer or transform (change) these energies.

Draw a cartoon of the initial state of the system, and draw a cartoon of the finial state of the system. Changes / physical process(es)?

The principle of the conservation of energy states that the total energy E_{sys} of a system of interacting objects remains constant. Interactions and processes inside the system can only transfer or transform the types of energy present inside constituents the system, not change the total amount of energy inside the system. This means that the final total energy of the system $E_{\text{sys,f}}$ is equal to the initial total energy of the system $E_{\text{sys,i}}$. (In this context “initial” and “final” are synonymous with “before” and “after”.) The mathematical expressions for this idea are:

$$E_{\text{sys,f}} = E_{\text{sys,i}} \tag{4.2}$$

$$E_{\text{sys,f}} - E_{\text{sys,i}} = 0\,\text{J} \tag{4.3}$$

$$\Delta E_{\text{sys}} = 0\,\text{J} \tag{4.4}$$

In cases when energy is added to, and/or removed from, the system, conservation of energy is written

$$\Delta E_{\text{sys}} = +E_{\text{added}} - E_{\text{removed}} \tag{4.5}$$

■**PICTURE:** Object as system, separate from Earth, versus Object and Earth as system

4.2 Temperature

The human senses are usually listed as sight, hearing, taste, smell, and touch. But the sense of touch actually has several different aspects to it: a sense of pressure, a sense of vibration, a sense of pain, and a sense of temperature. It is this sense of temperature that lets us know if something is "hot" or "cold".

Like all other human senses this sense of temperature is limited and subjective. Just as there are wavelengths of light we can not see (ultraviolet and infrared, for example), there are temperatures we can not experience. Just as our sense of taste can be confused by the order in which we eat things (like orange juice and toothpaste), our sense of temperature can be confused by exposure to extremes of temperature ("cold" water can feel hot if we just came in from $-30\,^{\circ}\mathrm{C}$).

Even though our senses of sight, taste and temperature are subjective and can be fooled, they do respond (and correspond) to the objective phenomena of electromagnetic radiation, chemical composition, and thermal energy.

4.2.1 Properties that Depend upon Temperature

Changing the temperature of an object can change its physical properties. The mechanical properties are the easiest to observe and measure quantitatively.

Size change

The primary physical property that changes with temperature is size. The size of solid objects increases with temperature; it expands. This effect is usually small (producing a strain in the range $10^{-6} < \epsilon < 10^{-4}$ for each Celsius degree of heating), but is measurable, and can have significant engineering affects.

■**PICTURE:** Expansion joints in a large bridge

An example of this are the necessity of expansion joints that are found on large bridges. For bridges with large spans (hundreds of metres) a strain of 10^{-4} corresponds to a change of several centimetres! The expansion joints (gaps) are there for the material to expand into. If they were not there, the bridge's expansion could break the material.

Liquids and gasses (if pressure remains the same) also expand with increasing temperature, meaning their density decreases. This principle is used to construct liquid thermometers.

Conversely, if the volume containing a liquid or gas is kept constant, the pressure they exert on the container will increase with increasing temperature, and decrease with decreasing temperature. It is this last fact that – through experiment – demonstrates that there is a "coldest temperature"; there is a *single temperature* at which the pressure of *any* gas becomes exactly zero. At that point there is no more thermal energy remaining to extract; there is no colder temperature. This is called the ***absolute zero*** of temperature.

Stiffness

The secondary property that changes is mechanical stiffness 3. In the case of solids usually the value of their Young's Modulus decreases as temperature increases, as do the values of their yield stress and maximum strain (it becomes easier to bend and break materials at higher temperature).

In the cases of liquids and gasses, as temperature increases they typically flow more easily (measured by the property called viscosity). Microscopically it becomes easier for their constituent molecules to slip past each other.

Phase Change

When the temperature is increased or decreased enough the object might change completely by melting or freezing, or by vaporizing or condensing. This would be what is called a *phase change*. After that, all its physical properties will be drastically different. These changes happen a temperature that is a characteristic of the material. Phase changes can hence be used to accurately determine when a specific temperature is achieved. (For example: if water *just* begins to freeze you know you have reached 0 °C.)

Other properties

After we've talked about *electrical* properties of materials (like electrical resistance) in chapter 6 we will see how temperature can also affect those properties.

The point of all these examples is that properties we can measure (like size, pressure, stiffness, or electric properties) are directly affected by changes in temperature. By measuring these changes, we can form an objective, quantitative measurement of the temperature.

4.2.2 Scales of Temperature

Celsius

To set a scale for measuring temperature a correspondence between temperature and some physical property must be established. In the case of the *Celsius scale*, the transition from solid to liquid water (melting) is defined to happen at $T = 0\,^\circ\mathrm{C}$, and the transition from liquid water to vapour (boiling) is defined to happen at $T = +100\,^\circ\mathrm{C}$. Temperatures between these two (like average body temperature $+37\,^\circ\mathrm{C}$) are of interest and are important in the human context. Temperatures below 0 °C (isopropyl alcohol freezes at $-89\,^\circ\mathrm{C}$) and above $+100\,^\circ\mathrm{C}$ (glycerol boils at around $+290\,^\circ\mathrm{C}$) exist and are measurable. But absolute zero is at $-273.15\,^\circ\mathrm{C}$ (exact), and there is no colder temperature.

Kelvin

The ***Kelvin scale*** of temperature is defined by setting $T = 0\,\mathrm{K}$ at the absolute zero of temperature, and then equating a change of one kelvin with a change of one Celsius degree:

$$\Delta T = 1\,\mathrm{C}^\circ = 1\,\mathrm{K} \tag{4.6}$$

(A *change* of one Celsius degree is written 1 C°. Note that the degree sign is *after* the unit symbol.) The kelvin, as a unit of temperature, is one of the *base units* of the SI system alongside the metre, second, and kilogram.

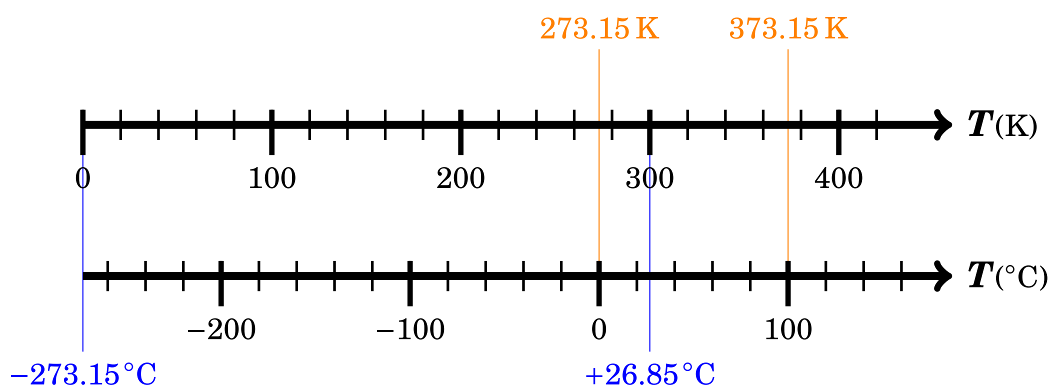
The important values to note are

$$0\,\mathrm{K} = -273.15\,^\circ\mathrm{C} \tag{4.7}$$
$$273.15\,\mathrm{K} = 0\,^\circ\mathrm{C} \tag{4.8}$$
$$293.15\,\mathrm{K} = +20\,^\circ\mathrm{C} \tag{4.9}$$
$$373.15\,\mathrm{K} = +100\,^\circ\mathrm{C} \tag{4.10}$$

Since one increment on the Kelvin scale is equal to one increment on the Celsius scale, the relation between the relation between the two is very simple:

$$T_\mathrm{C} = T_\mathrm{K} - 273.15\,^\circ\mathrm{C} \tag{4.11}$$
$$T_\mathrm{K} = T_\mathrm{C} + 273.15\,\mathrm{K} \tag{4.12}$$

(Don't bother memorizing this – I will give it to you, or you can look it up, if you'll need it.) This information is shown graphically, below.



Fahrenheit

There is another temperature scale that you have probably heard of: the *Fahrenheit* scale. Currently it is used only in the United States of America. But again, given that country's importance as our largest trading partner, we can not afford to completely ignore it.

The physical phenomena and numerical choices that were used to define this scale are a little unusual:

- There is a specific mixture of water, salt, and ammonia that goes to a specific fixed temperature by a chemical reaction; this was (initially) defined to be 0°F.
- The increment of the scale was adjusted so that the temperature at which liquid water freezes to ice was +32°F, and body temperature was +96°F (so that there was $64 = 2^6$ increments between the two).
- Experimentally, this made the temperature at which liquid water boils into steam near +212°F on this scale. The scale was further adjusted so that this became an exact number.

The history of its development is even *weirder* than that, but you get the idea.

The important values to know are

$$+32°\mathrm{F} = 0\,°\mathrm{C} \tag{4.13}$$

$$+212°\mathrm{F} = +100\,°\mathrm{C} \tag{4.14}$$

Consequently

$$1\,\mathrm{F}° = \tfrac{5}{9}\,\mathrm{C}° = \tfrac{5}{9}\,\mathrm{K} \tag{4.15}$$

This shows that one increment on the Fahrenheit scale is a smaller increment than one on the Celsius (and Kelvin) scale. The relation between the Fahrenheit and Celsius scales is linear, and a little algebra can show that

$$T_\mathrm{C} = \tfrac{5}{9}\left(T_\mathrm{F} - 32°\mathrm{F}\right) \tag{4.16}$$

$$T_\mathrm{F} = +32°\mathrm{F} + \tfrac{9}{5}\,T_\mathrm{C} \tag{4.17}$$

(Don't bother *memorizing* the values or formulas in this subsection. I will give them to you, or you can look them up, if needed.) Given this relation, here are a few correspondences:

$$-40°\mathrm{F} = -40\,°\mathrm{C} \tag{4.18}$$

$$0°\mathrm{F} = -18\,°\mathrm{C} \tag{4.19}$$

$$+68°\mathrm{F} = +20\,°\mathrm{C} \tag{4.20}$$

$$+100°\mathrm{F} = +38\,°\mathrm{C} \tag{4.21}$$

4.2.3 Thermal Equilibrium

4.3 Thermal Energy

We know from experience and experiment that temperature relates to energy. Specifically increasing the thermal energy of an object increases its temperature. But temperature is not the same as, and is *not equal to*, the thermal energy of an object.

4.3.1 Energy and Temperature

Transferring thermal energy into or out of an object changes its temperature. If we add an increment of thermal energy $\Delta E > 0\,\mathrm{J}$ its temperature increases $\Delta T > 0\,\mathrm{C}°$ proportionally. Similarly, if we remove an increment of thermal energy $\Delta E < 0\,\mathrm{J}$ its temperature decreases $\Delta T < 0\,\mathrm{C}°$ proportionally. The proportionality depends upon two factors: the *amount* and *type* of material.

If an amount ΔE of energy is required to warm an amount of mass m by an increment ΔT, then twice that amount of energy will be required to warm twice as much mass by the same increment. For this reason the amount of thermal energy required to achieve a specific change in temperature is proportional to the amount of mass whose temperature we are changing.

The remaining factor that relates temperature to thermal energy is the identity of the material. Different materials are "easier" to heat or cool than others. The heat capacity of water is $\mathscr{C} = 4\,184\,\mathrm{J/kg \cdot C°} \approx 4\,\mathrm{J/g \cdot C°}$. This can be compared to other materials like air

(1), aluminum (0.2), steel (0.5), wood (pine: 1.5), and brick (0.8) (all measured in units of $\mathrm{J/g \cdot C^\circ}$).

These relations are combined in a single mathematical expression. If the temperature of an object changes by an amount ΔT, then the amount of thermal energy ΔE transferred into ($\Delta T > 0\,\mathrm{C}^\circ$) or out of ($\Delta T < 0\,\mathrm{C}^\circ$) the object is

$$\Delta E = m\mathscr{C}\,\Delta T \tag{4.22}$$

where m is the mass of the object, and $\mathscr{C}$ is the heat capacity that's a property of the object's material.

Units of Energy

The Metric unit of energy is the ***joule***. It is defined in terms of mechanical interactions: the energy transferred by a force of one newton acting through a distance of one metre ($1\,\mathrm{J} = 1\,\mathrm{N} \times 1\,\mathrm{m} = 1\,\mathrm{kg \cdot m^2/s^2}$). relative to this unit the heat capacity of water has the value $\mathscr{C} \approx 4\,\mathrm{J/g \cdot C^\circ}$.

Another unit of energy was defined by the properties of water. A ***calorie*** is defined to the amount of energy required to increase the temperature of one gram of water by one degree Celsius. Measured relative to this unit the heat capacity of water is exactly $\mathscr{C} = 1\,\mathrm{cal/g \cdot C^\circ}$. This means that

$$1\,\mathrm{cal} = 4.184\,\mathrm{J} \tag{4.23}$$

$$0.239\,\mathrm{cal} = 1\,\mathrm{J} \tag{4.24}$$

Remember it this way: the calorie is a slightly larger unit of energy than the joule, and so energies measured in calories will have smaller numbers. (This unit is sometimes referred to as the "thermodynamic calorie".)

Unfortunately there is a *different* commonly-used unit that is also called a "calorie". This is the *nutritional calorie*. It is denoted by "Cal", with the leading "c" capitalized. A nutritional calorie is a thousand thermodynamic calories:

$$1\,\mathrm{Cal} = 1\,000\,\mathrm{cal} = 1\,\mathrm{kcal} = 4\,184\,\mathrm{J} = 4.184\,\mathrm{kJ} \tag{4.25}$$

This unit is useful in the human context because: the masses (of water in the body) involved in metabolic processes are on the order of tens of kilograms; and the typical nutritional energy content of a single meal is on the order of hundred of thousands of joules. (Be very careful reading and writing energies in terms of "calories" – know which one is being used!)

4.3.2 Transfer of Thermal Energy

How can thermal energy be transferred?

"Heat"

For historical reasons an amount of thermal energy that is in the process of being transferred is called ***heat*** and is symbolized by the letter Q. Be careful with the distinction, that thermal energy *in* the object is just thermal energy, but that thermal energy being *transferred* to or from an object is "heat". Alternatively, heat is thermal energy in transit.

Conduction

When two objects of different temperature are in contact thermal energy flows from the hotter object to the cooler object until they are at the same temperature. This is an experimental fact. But what's happening that makes this so?

If two objects touch each other thermal energy can be transferred from one to the other. At the atomic scale it is the molecules in one object (that are wiggling around because of their thermal, incoherent kinetic energy) colliding with the molecules in the other object. These collisions transfer (incoherent) kinetic energy; that is, they transfer thermal energy.

If the temperatures are the same, the transfers are equal, and the net (total) transfer is zero (each gains from the other what it loses). But when the temperatures are different, more thermal energy is transferred from the hotter towards the colder than from the colder towards the hotter. As their temperatures approach being equal the net transfer decreases towards zero.

Direct Contact versus Conductive / Insulating separator. (This will be explored quantitatively in subsection 4.3.4.)

The word "flow" is used to talk about the transfer of thermal energy. Have it clear in your mind that no material is moving from one object to the other. The word "flow" is used because of the similarity in the *mathematics* that describes this transfer with the math that describes gasses flowing diffusively. We will speak about the "flow of thermal energy" even though no matter is moving or being transferred.

Convection & Advection

If a warm object is touching a cooler fluid material (a liquid or a gas), thermal energy transferred from the object to the fluid can be carried away from the object by the fluid's flow.

When a fluid's temperature increases it expands, so its density decreases relative to the surrounding fluid. The difference in buoyant forces will cause this warming volume of fluid to rise and the nearby cooler fluid to sink. This motion will carry away some of the object's thermal energy, and will also bring a new quantity of cooler fluid back into contact with the object. This can start a larger flow in the volume of the fluid, which helps keep the process going. This type of fluid motion driven by the transfer of thermal energy is referred to as ***convection***, and sometimes more specifically as *convective cooling*.

When the transfer of thermal energy is what drives the fluid flow it is referred to as convection. But when it is an external cause forcing the flow of the fluid (like wind, or a fan) is referred to as ***advection***. In deep winter the phenomena of *wind chill* is an example of advective cooling. The central processing unit (CPU) of a computer usually has a fan mounted on top of it aimed to drive cooler air into contact with the CPU.

But advection can also bring thermal energy *to* an object. So-called "convection ovens" are actually using advection to drive hot air over the object (in the case of an oven, the object is the food being cooked). If the food is cooler than the air being advected around it, then the food takes thermal energy from the air; the air gets cooler and the food gets hotter. Advection then replaces the cooled air with new, hotter air, driving the process of cooking forwards.

In both cases Convection and Advection moves the medium into contact with the object, increasing the difference in temperature at the surface of contact, thereby increasing conduction, driving the process of thermal energy transfer.

Radiation

The word "radiation" means energy being transported from place to place without a transport of material. Strictly speaking "sound" (the topic of chapter 5) is acoustic radiation, "light" is electromagnetic radiation, and the heat that you feel from a hot stove-top is thermal radiation.

Visible light is *electromagnetic radiation*, but is not the full spectrum.

Light in the infra-red portion of the spectrum transfers thermal energy. This is the "heat" that you feel at a distance from a large fire, or a heating element on a stove.

Evaporation & Condensation

Liquid at the surface of a hot object can absorb thermal energy from the object and transform to its vapour phase, which then leaves the object's surface, taking that thermal energy with it. This is referred to as *evaporation*, and sometimes more explicitly as *evaporative cooling*.

The reverse of this process is *condensation*. Vapour in the air surrounding a cold object can transform to its liquid phase when it lands on the surface. During this transformation from vapour to liquid the vapour must *release* thermal energy (since this is the reverse of boiling where the fluid absorbs thermal energy). This thermal energy is then transferred to the object, increasing its temperature.

4.3.3 Transformation to Thermal Energy

The transformation of other forms of energy into thermal energy.

Dissipative interactions: friction and fluid drag.

4.3.4 Power

Recall from sub-section 0.1.6 the defining property of a *rate*:

$$\text{change} = \text{rate} \times \text{time} \tag{4.26}$$

The rate at which energy is transferred or transformed is called ***power***:

$$\Delta\,\text{Energy} = \text{Power} \times \text{Time} \tag{4.27}$$

(Remember that the prefixed symbol "Δ" means "change in ... ".) Mathematically we write

$$\Delta E = P \times \Delta t \tag{4.28}$$

$$P = \frac{\Delta E}{\Delta t} \tag{4.29}$$

With energy measured in joules and time in seconds the units of power are *watts*: $1\,\text{W} = 1\,\text{J/s}$. (Note carefully!: the "t" in the denominator "Δt" is a lower-case "t" denoting *time*, not an upper-case "T" denoting temperature.) In that expression ΔE is the amount of energy that is being transferred or transformed, and P is the rate at which that process if happening.

In the case of thermal energy being transferred, the expression for power is

$$P = \frac{Q}{\Delta t} \tag{4.30}$$

where Q is the total thermal energy transferred during the time interval Δt.

Conduction

In the case of conduction of thermal energy through a boundary between two objects (or systems) is given by the expression

$$P = \frac{Q}{\Delta t} = \mathscr{K}\frac{A}{L}\left(T_2 - T_1\right) \tag{4.31}$$

This equation tells us the rate at which thermal energy is transferred into object 1 from object 2 when they are separated by a boundary. The greater the area A of contact, the faster thermal energy will be transferred. The greater the thickness L of the boundary, the slower thermal energy will be transferred. The physical property of the boundary material that governs how fast heat can flow through it is its *thermal conductivity*, symbolized by $\mathscr{K}$.

When object 2 is hotter than object 1 ($T_2 > T_1$) the power calculated by the above equation is positive $P > 0$ W, which means that thermal energy is flowing into object 1 from object 2.

Sometimes the expression above is taken in the context of an enclosed system and its surrounding environment:

$$P = \mathscr{K}\frac{A}{L}\left(T_{\mathrm{env}} - T_{\mathrm{sys}}\right) \tag{4.32}$$

where $T_{\mathrm{env}} - T_{\mathrm{sys}}$ is the temperature difference across the boundary (between the system inside the boundary, and the environment outside the boundary). As before when this quantity is positive, it means that thermal energy is being transferred *into* the system, while negative would mean that thermal energy is *leaving*.

4.3.5 Achieving Thermal Equilibrium

It is a physical Law that thermal energy will spontaneously flow from a hotter object to a colder object. The thermal energy lost by the hotter object is gained by the cooler object. As time goes on, the temperature of the two objects tend towards a single common temperature that is between their two initial temperatures.

Experiment finds that the energy lost by one object equals the energy gained by the other:

$$\Delta E_A + \Delta E_B = 0\,\mathrm{J} \tag{4.33}$$

This is a manifestation of the principle of the conservation of energy.

The equation above shows that if, for example, object A cools ($\Delta E_A < 0$ J as thermal energy leaves) then object B will warm ($\Delta E_B > 0$ J as thermal energy enters). With sufficient time the temperature of each object will change and approach a common value T_f. The details are as follows:

$$\Delta E_A + \Delta E_B = 0\,\mathrm{J} \tag{4.34}$$

$$m_A \mathscr{C}_A\, \Delta T_A + m_B \mathscr{C}_B\, \Delta T_B = 0\,\mathrm{J} \tag{4.35}$$

$$m_A \mathscr{C}_A\, (T_\mathrm{f} - T_{A\mathrm{i}}) + m_B \mathscr{C}_B\, (T_\mathrm{f} - T_{B\mathrm{i}}) = 0\,\mathrm{J} \tag{4.36}$$

$$(m_A \mathscr{C}_A + m_B \mathscr{C}_B) T_\mathrm{f} - (m_A \mathscr{C}_A\, T_{A\mathrm{i}} + m_B \mathscr{C}_B\, T_{B\mathrm{i}}) = 0\,\mathrm{J} \tag{4.37}$$

The result is that the common final temperature of the two objects will be

$$T_\mathrm{f} = \frac{m_A \mathscr{C}_A\, T_{A\mathrm{i}} + m_B \mathscr{C}_B\, T_{B\mathrm{i}}}{m_A \mathscr{C}_A + m_B \mathscr{C}_B} \tag{4.38}$$

This weighted-average of the two initial temperatures will always be between the two. (The combination of factors $m\mathscr{C}$ is referred to as the *thermal mass* of the object.)

In the case that the two objects are the same material, the value of the heat capacity is a common factor in the numerator and denominator which cancels. In this case the final temperature is given by the simpler expression

$$T_\mathrm{f} = \frac{m_A\, T_{A\mathrm{i}} + m_B\, T_{B\mathrm{i}}}{m_A + m_B} \tag{4.39}$$

This will used in cases where, for example, we mix two volumes of water together.

4.4 Kinetic Energy

If you open up any standard physics textbook one of the first forms of energy discussed is always kinetic energy. You may even remember it from high school:

$$K = \tfrac{1}{2} m v^2 \tag{4.40}$$

This formula says that: a larger mass has more kinetic energy than a smaller mass traveling at the same speed; and, an object traveling a larger speed has a larger amount of kinetic energy.

The expression $\frac{1}{2}mv^2$ also shows the units that compose a joule. With mass in kilograms and speed in metres per second, a joule is defined as

$$1\,\text{joule} = 1\,\mathrm{J} = 1\,\mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s}^2 \tag{4.41}$$

It is relative to this definition that the heat capacity of water ($4\,182\,\mathrm{J/kg\cdot C^\circ}$) has its value.

Kinetic energy is the energy of motion: of something with changing position, orientation, or shape. To get an object moving I have to make an effort and transfer some energy to it. To stop an object from moving I have to make an effort and transfer some energy out from it, or make it interact with some other object / objects that will absorb some of its energy.

There are a few different ways that something might move:

- it might move along a straight line from one place to another
- it might turn without going anywhere
- it might wiggle or bend or move back-and-forth repeatedly (vibrate or oscillate)

- its atoms / molecules might be jiggling without breaking their mutual bonds

The energy associated with motion from place to place is called *linear* kinetic energy. The energy associated with turning is called *rotational* kinetic energy. The other forms of motion can become very complicated to describe, so we will begin with just linear and rotational for now.

4.4.1 Linear Kinetic Energy

Motion is about the position of pieces of material changing. The most basic type of motion is *linear motion*. In this type of motion the object, as a whole, moves from place to place along a straight line. Each piece of the object moves in the same direction and travels the same distance.

The measurement of the rate of change of position is *speed*. The vector that points in the direction of the object's motion, and whose magnitude is the speed, is called the ***velocity***. In the case where linear motion is along the x-axis, the x-component of the object's velocity is

$$v_x = \frac{\Delta x}{\Delta t} \tag{4.42}$$

For motion in three dimensions, there are similar expressions for v_y and v_z. The object's speed is the magnitude of the velocity:

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} \tag{4.43}$$

If the object is composed of N pieces, then the kinetic energy K of the object as a whole is the sum of the kinetic energies K_n of its pieces. (The subscript "n" on the quantity K_n labels which of the N pieces we are talking about.)

$$K = \sum_{n=1}^{N} K_n = \sum_{n=1}^{N} \tfrac{1}{2} m_n v_n^2 \tag{4.44}$$

If every piece of the object is moving with exactly the same speed ($v_n = v$ for all n) then

$$K = \sum_{n=1}^{N} \tfrac{1}{2} m_n v_n^2 = \tfrac{1}{2}\left(\sum_{n=1}^{N} m_n\right) v^2 = \tfrac{1}{2} m v^2 \tag{4.45}$$

where $m = \sum_{n=1}^{N} m_n$ is the sum of the masses of the pieces, which is the total mass of the object.

Forces affect Linear kinetic energy (see subsection 4.4.5 below on the subject of "Work").

4.4.2 Angular Kinetic Energy

When an object rotates or turns about a pivot, the angle that specifies the direction it is facing changes: $\Delta\theta \neq 0\,\text{rad}$. In this type of motion it is possible for the object to be moving but going anywhere. Each piece of the object circles about the axis of rotation, while the object, as a whole, does not change its location ($\Delta x = 0\,\text{m}$).

Angular speed

The *rate* at which this angle changes is called the ***angular speed***:

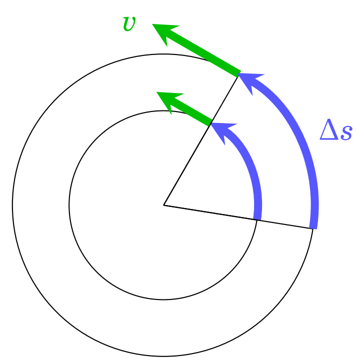
$$\omega = \frac{\Delta\theta}{\Delta t} \tag{4.46}$$

(The symbol for angular speed is the Greek letter *omega* ω, which is written like a curvy lower-case double-u.) The units of angular speed are radians per second. If the angles used to describe the object's orientation are measured in degrees, then do not forget to convert to radians before using any of the formulas in this subsection.

Two other common units used to measure the rate of rotation are revolutions per second, and revolutions per minute (rpm). One "revolution" is a complete rotation: $1\,\mathrm{rev} = 2\pi\,\mathrm{rad}$.

▲**FIX:** RPM sample calculation / Example

Sum of contributions



When an object rotates or turns about a pivot, each piece is moving on a circle. A piece that is a distance r from the axis of rotation moves a distance $\Delta s = r\,\Delta\theta$ (by definition of an angle measured in radians).

If the object turns an angle $\Delta\theta$ in a time Δt, then the pieces that are further from the axis of rotation travel faster:

$$r_B > r_A \tag{4.47}$$

$$r_B\,\Delta\theta > r_A\,\Delta\theta \tag{4.48}$$

$$s_B > s_A \tag{4.49}$$

$$(s_B/\Delta t) > (s_B/\Delta t) \tag{4.50}$$

$$v_B > v_A \tag{4.51}$$

This is important because it means that each piece that's a different distance from the axis of rotation is traveling a different linear speed, and each piece will make a different contribution tot he total kinetic energy. Consequently the kinetic energy due to rotation is not proportional to the total mass of the whole object. The total kinetic energy is

$$K = \sum_{n=1}^{N} \tfrac{1}{2} m_n v_n^2 = \tfrac{1}{2} \sum_{n=1}^{N} m_n (r_n \omega)^2 = \tfrac{1}{2} \left(\sum_{n=1}^{N} m_n r_n^2 \right) \omega^2 \tag{4.52}$$

The sum inside the brackets ($\sum m_n r_n^2$) is *not* the mass of the object. What is it?

The Moment of Inertia

The factors of m and r^2 in each term of the sum combines information about how the mass of the system is distributed across its shape relative to the axis of rotation. This quantity is called the ***moment of inertia***:

$$\mathscr{I} = \sum_{n=1}^{N} m_n r_n^2 \tag{4.53}$$

$$K_{\text{angular}} = \tfrac{1}{2} \mathscr{I} \omega^2 \tag{4.54}$$

Torques effect Rotational kinetic energy.

▲**FIX:** EXAMPLES: Include table of moments for simple gemometries, paired with numerical examples of values (ie. equal mass but different shape). Match with examples that can be compared in the Labs.

4.4.3 Oscillation and Vibration

If the shape of the object is not constant. Waves in chapter 5. Normal modes.

Similar to rotational motion, in this category the object as a whole does not move to a new location.

4.4.4 Thermal Kinetic Energy

The atoms and molecules that make up any object are all continuously, randomly, wiggling. This is an experimental fact. Each piece is wiggling in a way that is independent of how the other pieces are wiggling. The average speed of each piece (as it wiggles back-and-forth) is dependent upon the temperature of the object: the higher the temperature, the faster the wiggling; the lower the temperature the slower the wiggling. The limits to this kind of motion are, of course: if the temperature is too high, then the molecules that make up the object break away from each other and the object melts or vaporizes; the lowest temperature would be when the molecules stop moving entirely, which would happen at absolute zero of temperature $T = 0\,\text{K}$.

In contrast to the cases of Linear or Rotational motion (where the object moves coherently as a whole) thermal kinetic energy is the incoherent motion of the pieces of the object (where each of the pieces move at different speeds and in different directions, independently of each other).

Friction transforms coherent kinetic energy into incoherent kinetic energy. Microscopic collisions between the tiny piece of rough surfaces.

4.4.5 Work

Work is Force times Displacement $W = Fd$.

Component of force that does work.

Component of force that does no work.

It is only the portion of the force that is parallel to the displacement that transfers energy to the object, that does work:

$$W = F\cos\theta\, d \tag{4.55}$$

(where θ is the angle between the force $\vec{F}$ and the displacement d). When the displacement is along the x-axis the expression above can be written

$$W = F_x\,\Delta x \tag{4.56}$$

Note on units: $\mathrm{N} \times \mathrm{m} = \mathrm{kg}\cdot\mathrm{m/s^2} \times \mathrm{m} = \mathrm{kg}\cdot\mathrm{m^2/s^2} = \mathrm{J}$. Also: $\mathrm{N} = \mathrm{J/m}$.

Work and Kinetic Energy

When a force is applied to an object, and that object's motion is effected, we say that *work has been done to the object*. The symbol for work is W. The work-kinetic energy theorem of mechanics states that

$$\Delta K = W \tag{4.57}$$

Work is the amount of energy that is transferred to (or transferred from) and object by an interaction, manifested as an applied force. (This definition excludes the transfer of *thermal* energy.) The portion of the force that does work is the portion that is parallel to the object's motion. If the object moves along a distance d, and the angle between that distance and the applied force is θ, then the work done by that force is:

$$F\,d\,\cos\theta = K_\mathrm{f} - K_\mathrm{i} \tag{4.58}$$

In the case where the object moves along the x-axis this equation is written

$$F_x\,\Delta x = \Delta K \tag{4.59}$$

where $\Delta x = x_\mathrm{f} - x_\mathrm{i}$ is the change in the object's position, and $\Delta K = K_\mathrm{f} - K_\mathrm{i}$ is the change in the object's kinetic energy. This equation is the link between force and energy.

Angular work

Work is "force times displacement". For linear motion, the displacement is a change in position, like Δx. but for angular motion, the "displacement" is a change in angle $\Delta\theta$. Force effects linear motion, but torque effects angular motion. For these reasons, the ***angular work*** done by torque is

$$W = \tau_z\,\Delta\theta \tag{4.60}$$

Because of this, we have that

$$\tau_z\,\Delta\theta = K_\mathrm{f} - K_\mathrm{i} \tag{4.61}$$

where we must be careful to remember that the form of kinetic energy here is the *angular* kinetic energy $K = \frac{1}{2}\mathscr{I}\omega^2$.

Note: Torque is not Work

Work is not torque, and torque is not work. Compare $F\,d\cos\theta$ (the definition of work) with equation 2.6 $F\,d\sin\theta$. Force *along* the displacement of the object does work. (The cosine finds the component parallel to the distance.) Force *around* a pivot applies a torque. (The sine finds the component perpendicular to the distance from the pivot.) In both cases there is a force and a distance. But in the case of torque the distance from the pivot is not changing and no work is done along that direction.

Another detail to note is the units of the quantities involved. The angular displacement $\Delta\theta$ is measured in radians. Remember that $1\,\text{rad} = 1\,\text{m}/1\,\text{m} = 1$ is actually just a pure number, without units. (We write "radians" next to an angular quantity more as a reminder than anything else.) $\text{N} \times \text{m} = \text{kg} \cdot \text{m/s}^2 \times \text{m} = \text{kg} \cdot \text{m}^2/\text{s}^2 = \text{J}$.

Angular work and Stability

Work required to counteract sway. Displacement of center of mass relative to base of support (linear displacement relative to surface versus angular displacement relative to "pivot".)

▲**FIX:** EXAMPLE?

4.4.6 Power

Power measures the rate at which energy is transferred or transformed. In the case of kinetic energy

$$P = \frac{\Delta K}{\Delta t} \tag{4.62}$$

Efficiency

What fraction of the total energy used actually accomplished work.

4.5 Elastic Potential Energy

You know that it takes effort to stretch an elastic band. You have to apply forces to the ends of the elastic to separate the ends: you apply a force across a distance, hence doing work to the elastic. If you stretch and then hold an elastic, your effort (the work that you did to stretch it) is stored in it as *elastic potential energy*. This energy you can get back as kinetic energy if you let the elastic launch itself.

4.5.1 "Potential"?

The energy of motion, which you can see, versus the forms of energy that you can't see.

Any form of energy that can be transformed back into kinetic energy is, for historical reasons, referred to as being *potential* energy (is in "it might, *potentially*, become energy we can see").

Since the agent of transformation is interaction in the form of a force, we HERE.

There are other forms of potential energy (gravitational and electrical being the most important ones), and we will discuss those in section 4.7.

4.5.2 Elastic Force

Our old friend from subsection 3.3.2: Hooke's law

$$F = kx \tag{4.63}$$

To know which way this force acts you must be very careful with identifying the object.

4.5.3 Energy stored in Deformation

The area under the stress-strain curve. Area under the force–displacement, and the work done.

The maximum amount of energy just before failure. Relate to biomechanics and ***injury***.

4.6 Pressure

Imagine an inflated balloon. The gas inside is exerting a pressure on the skin of the balloon, pressing outwards. This pressure is a force distributed across the inner surface of the balloon.

Imagine now that nothing was constraining the size of the balloon; that this force could continue to push on the skin of the balloon, and the balloon could continue to expand. Two things would happen:

- The pressure would do work as this force acted across the distance traveled by the skin of the balloon, ($W = F\Delta x = PA\,\Delta x$).
- The pressure would decrease as the volume increased (remember the ideal gas law $P = nRT/V$).

Work would be done by the pressure as the balloon expanded, but should be finite because the pressure would be decreasing, and would approach zero.

Considering the work done, where does this energy come from? The answer is that the pressure is a measure of the energy content U of the gas. Detailed calculation can show that

$$P = cU\big/V \tag{4.64}$$

(where the coefficient $1/3 \leq c \leq 2/3$ is a property that depends upon the type of gas). This says that pressure is a measure of the energy density (J/m^3) of the gas that can do work. Conversely this says that pressure is a measure of the work that had to be done to compress the gas into the volume V.

▲FIX: Pressure as it relates to energy density, and fluid flow.

4.7 Interactions

Interactions between objects can transfer and transform the different forms of energy. When describe a collection of interacting objects as the system, their interactions are also inside the system. When this is done we find that a new category of energy is present: the category of *potential energy*.

4.7.1 Work versus Potential Energy

The relation between work and kinetic energy we have seen before:

$$\Delta K = W \tag{4.65}$$

We would use this form of energy equation when the system is a single object, and the only form of energy we need to consider is the object's kinetic energy. In this context there is a force whose cause is outside the "system" that is changing the system's energy (the kinetic energy).

Introduced in section 4.5 we the idea of forms of energy that could, through interaction, be transformed and manifest as kinetic energy. Energy of this type was categorized as ***potential*** energy.

4.7.2 Gravitational Interactions

Gravity is the attractive force that each mass exerts on every other mass. This interaction has three main characteristics:

- It is only attractive. Gravity never causes a mass to repel another mass; it only ever attracts.
- It is extremely weak. Only masses on the order of the size of moons and planets exert significant gravitational forces on other masses.
- The strength of the gravitational attraction decreases with separation between the masses. Mathematically $|F_\mathrm{G}|$ decreases with separation as $1/r^2$.

The decrease with distance is why the moon (whose mass is 1/81 of Earth's mass) at a distance of $3.85 \times 10^{+8}$m causes the ocean tides, but Jupiter (whose mass is 318 times Earth's mass) at a distance of $7.78 \times 10^{+11}$m (about 2000 times further away) exerts almost no measurable force on Earth-bound objects. As mentioned previously in subsection 1.2.4, the only gravitational force that will matter for you is Earth's field.

Since gravity is attractive between masses, it would take work to separate them. This might remind us of when we pull on opposite ends of a spring or elastic, where it takes work to pull the ends further apart.

Context: near the Earth's surface. Changes in height.

$$\Delta U_\mathrm{G} = mg\,\Delta h \tag{4.66}$$

Falling. Rising and Falling. Trajectories.

Context: not near the Earth's surface. Orbits.

4.7.3 Contact Interactions & Work

Pushing, pulling. Compression or Tension ("pressure" vs stress terminology). Friction. Collisions. Transformation of Kinetic to... Bouncing, bending, breaking.

4.7.4 Electromagnetic Interactions

(Covered in much more detail in later chapters.)

Electric charges. Interacting charges.

Electric dipoles. Interacting dipoles.

Electric currents. Electrical Potential Energy. Circuits.

Interacting currents. Magnetic dipoles. Interacting dipoles.

Chapter 5

Waves

In your *Electrotherapy* course you will learn about using ultrasound equipment for therapy purposes. To prepare for that, we need to help you understand how to think about and analyze *waves*.

When you read the word "wave" you might imagine a large wave on the ocean, perhaps with a little surfboarder riding it. Yes, that is an example of a wave, but there are many other kinds of waves.

There are waves that we can see. Waves on the surface of water, and waves moving along a stretched string or rope are examples of what are called *mechanical waves*. These waves are referred to as "mechanical" because it is the motion of material (atoms and molecules) that make up the parts of the wave. (The exception of wave you can see but that is not mechanical is *light*.)

There are waves that we can't see. Sound is the best example of this. Obviously you hear sound, you don't see it, but you know that air is a material. If you wave a large fan with your hand, you can feel resistance; when the wind blows, you can feel it move your hair or clothes. Sound is a mechanical wave in this material. (Experiments can be done to prove that sound is a wave, and we might have some time to explore these during the semester, but it will not be our focus.) Understanding and being able to analyze sound waves in air and in water are the primary goal of this chapter.

Electrical signals, in wires and across wireless, are also waves that you can not see. These waves are *not* sound. These waves are not carried by material moving around. Understanding what electrical waves are will be the focus of chapter 6.

It may seem overwhelming that there are so many different kinds of waves, but we are lucky. All these seemingly different kinds of waves can be described using the *same concepts* and using similar mathematics. To help us grasp the concepts this chapter will focus on mechanical waves, specifically waves on a rope, and on sound waves in air and in water.

5.1 What is a Wave?

The examples mentioned in the introduction are meant to help you see that you already do know something about waves. To go forward we will now investigate how to think *quantitatively* about waves. We will first identify the physical elements and motions of a wave. Once we have done that we will find that a wave is not an object: a wave is *not material* moving from one place to another, it is *energy* moving from one place to another.

5.1.1 The Three Elements of a Wave

Imagine you are holding one end of a long rope that is stretched between you and another person. To make a wave travel across the rope from you to the other person you would take your end and wiggle it up and down. The wiggles you just made would then *appear* to move along the stretched rope towards the other person. If you were to look closely at pieces of the rope in the segment between the two of you, you would see that the pieces of the rope move up and down as those wiggles passed.

There is no part of the rope that moves from your end to their end. The piece that you are holding never leaves your hand. After you stop wiggling your end, and the wave is finished, the rope is back to how it was before you started wiggling. Each piece of the rope is back in its original position. (This is in contrast to a situation like throwing a ball back-and-forth, where the material of the ball changes its location.)

So that we can discuss quantitatively and precisely what is happening we need to introduce some terminology. A wave has these three elements:

- the medium
- a source, that produces a disturbance of the medium
- propagation of the disturbance across the medium

In the example above: the stretched rope is the *medium*; you wiggling your end of the rope are the *source*; and those wiggles traveling towards the other person is a physical process called *propagation*. These concepts are very general, and apply to waves of all types.

For most of these first few sections of this chapter our development will focus on the example of waves traveling on a rope. Just remember that the concepts and mathematics for waves traveling on a rope apply equally well to other types of waves. Recall this in section 5.5, when we turn to studying *sound waves*.

the Medium

Different portions can have different deformations.

Examples of media. Gas, liquid, solid, biological. (Light: medium is *space* itself!)

Importance of the equilibrium configuration. What constitutes a disturbance. The idea of the “displacement” relative to the equilibrium.

the Source

Examples of sources; things that can cause “disturbances” of the equilibrium.

Properties of a periodic source: amplitude and rate of displacement. For mechanical waves the energy content (and through the rate, the power) will be determined by what *work* the source does to the material of the medium.

Propagation

The interacting pieces of the medium that create propagation. Details in the section 5.2.

5.1.2 A Wave is *not* an Object

The wave is not an object. Notice how the piece of rope that you are holding in your hand does not move away from you and travel towards the other person. The piece in your hand remains in your hand. The material of the medium does wiggle as the wave travels from you to them, but after the wave is done the rope is as it was before; the rope has not traveled, and the piece that was in your hand is still in your hand. So *what* moved along the rope?

Transfer of energy across the medium, not material. Keeping track of the energy (kinetic and elastic). Dissipation of energy (see Attenuation, below).

5.1.3 Graphs of Waves

The displacement from equilibrium is a function of *two* variables: when you look at the wave, and which piece of the medium you choose to watch. The when is obviously time, the variable t. The choice of which piece of the medium you watch is specified by a value of x (position) along the medium.

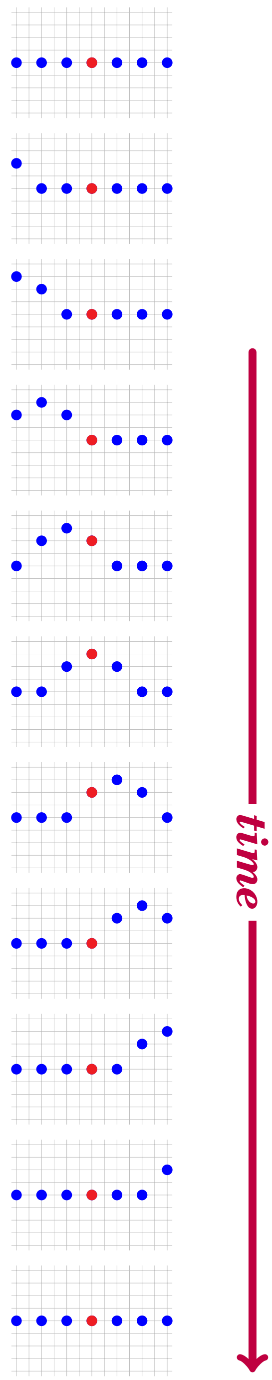
Note : Graphs of Waves

When reading a graph of a wave first look at the axes of the graph and note the units of the axes. Is the horizontal axis time, or is it position? This will tell you which type of graph you are reading, and what information you can hope to gain from reading it.

Plotting the displacement as a function of position x is like taking a photograph. This graph will be a representation of the wave, but only at a single, specific instant t in time. To choose a value of x on that graph is to choose a specific piece of the medium at that time. Reading the graph at that choice of x then tells you its displacement from equilibrium at the time t that the graph represents.

Plotting the displacement as a function of time t is watching the displacement of a specific piece of the medium. This graph is a representation of the motion of a specific piece of the medium whose equilibrium position is x. This graph is the kind of position-time graph that you have seen before in your high school physics course.

We will sometimes need both representations because the displacement as a function of time (the motion of the pieces of the medium) is what the source controls, while the displacement as a function of position (the "photograph" at a specific instant) is what we would naturally think of as the "wave". Being able to read graphs of these types and extracting information from them are skills we will practice in the exercises.


time

5.1.4 Types of Waves

In the category of mechanical waves, waves can be classified by the *way* in which the medium is displaced from its equilibrium. The categories follow those we saw in chapter 3 when we studied deformation: Compression, Tension, Shear, and Torsion. To make sense of these we will now study a specific example of a wave: the *pulse*.

5.2 Pulses

The medium in equilibrium. Producing a single pulse.

5.2.1 Motion of the Medium

A qualitative examination of the physics of wave propagation. The way in which the pieces interact to produce this propagation. The physics of how the properties of the medium determine the speed of wave propagation.

The idea of *displacement*.

[[diagram: time series of a pulse moving across a chain clearly showing the motion of each link in sequence.]]

Distinguishing between the wavespeed and the particle speed.

Location of the pulse as a function of time

$$x_{\mathrm{f}} = x_{\mathrm{i}} + v\,\Delta t \tag{5.1}$$

where v is the wavespeed of the medium. (The last term is $+v\,\Delta t$ if the wave is traveling towards the right, and would be $-v\,\Delta t$ if the wave was traveling towards the left.) This equation has the same mathematical form as the description of an object moving at a constant speed. Though it looks the same, it not describing the position of an object. It is describing the position of the disturbance as a function of time. Remember that the disturbance is not an object; the disturbance is just that portion of the medium that is not in its equilibrium. The remarkable fact about this equation is that it is true *independent of the shape of the pulse*.

Proof of wavespeed constancy???

The position of the particles of the medium do not follow any simple mathematical expression. Their motion is determined entirely by the shape of the pulse. Since the pulse may have almost any shape at all the motion of the particle can not be written out as a formula that is independent of the wave.

5.2.2 Mechanics of a Wave

A close look at the motion of the pieces of the medium, and their connections. Forces causing the changing motion (oscillation) of the particles.

Properties of the Medium: Density and Elasticity. How difficult is it to get the medium moving (mass from density). How easy is it for a disturbance to propagate to neighboring pieces (elasticity).

Energy is the next section. Kinetic energy of each piece. Elastic energy (or energy density in Pressure) between pieces.

Do we need to discuss superposition???

5.2.3 Categories of Wave

Forces between the particles of the medium are the cause of the wave's propagation. We know from chapter 3 that forces exerted on a material cause deformation. If you recall from chapter 3 there were different categories of deformation: tension or compression, shear, and torsion. Each category of deformation has corresponding category of wave type.

Transverse waves. In the case of a rope (or other medium that is much thinner than it is long), fluctuations in position or elastic deformation and hence tension in the medium. The tension connecting the particles of the medium propagate the wave. In the case of a solid (like the ground), shear stress propagates the wave.

Longitudinal waves. In the case of a fluid (gas or liquid), fluctuations in pressure in the medium. In the case of a solid, compression and tension.

There are also torsional waves, but we are not going to study them in any detail. If we were studying to engineer vehicles then we would need to know about these types of wave. But, in the context of Physiotherapy Technology, we don't need to spend any time on them.

Electromagnetic waves. Light (not covered in this course), and Electrical Current (chapter X).

Here are a few examples that are really, *really* outside this course: Gravity Waves; Traffic waves; Matter Waves in Quantum Mechanics; Chemical and Biological waves. Ask me about them, if we have spare time!

5.3 Energy Transport

A close look at the motion of the pieces of the medium, and their connections. Kinetic energy of each piece. Elastic energy (or energy density in Pressure) between pieces.

Transfer of energy across the medium, not material.

Keeping track of the energy (kinetic, and elastic or pressure).

Dissipation or transformation of energy (see Attenuation in subsection 5.3.4).

5.3.1 Energy in a Wave

In the case of mechanical waves the source does work to the material to set it into motion. The work transfers energy into the medium. As the wave propagates across the medium it carries this energy. What forms of energy are present in the wave? The answer will depend upon the medium, but in the case of mechanical waves there will always be a contribution of kinetic energy. This is because the pieces of the medium are wiggling as the wave passes across them.

Presence of elastic energy, or energy stored as pressure in the gas/liquid.

5.3.2 Work done to the Medium

In mechanical waves work must be done to the material of the medium to cause its motion.

Relation between the rate of oscillation (the frequency) and the rate of energy (power) being put into the medium.

The power transport by the wave. Careful to distinguish between amount (energy) and rate (power).

5.3.3 Waves in 2 & 3 Dimensions

A wave traveling along a string or rope is the easiest to see, draw or think about. That is an example of a wave traveling along a one-dimensional medium. Waves that travel on the surface of water are waves traveling in two dimensions. Sound waves in air are waves traveling in three dimensions.

The idea of *wave-front*.

The spreading of energy (), qualitatively.

Defining intensity?

$$\text{Intensity} = \frac{\text{Power}}{\text{Area}} \tag{5.2}$$

In the case of spherical wave-fronts, the inverse-square law in three dimensions:

$$I = \frac{P}{4\pi r^2} \tag{5.3}$$

5.3.4 Attenuation

The absorption or dissipation of energy (attenuation). The attenuation of mechanical waves due to internal friction or viscosity.

The exponential decrease with depth of the intensity.

Dependence on material and frequency.

(Discussed further in subsection 5.6.3.)

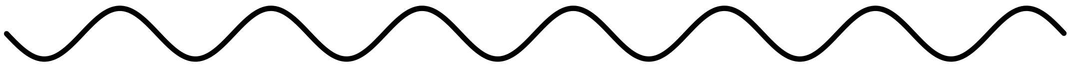
5.3.5 At a Boundary: Reflection & Transmission

Waves at a boundary between media: reflection or transmission.

5.3.6 Across a Boundary: Refraction

5.3.7 Around Obstacles: Diffraction

5.4 Periodic Waves



5.4.1 Properties of the Source

Amplitude. In the case of mechanical waves the amplitude relates to the physical distance that the source moves as it disturbs the medium. Be careful of the factor-of-two mistake: amplitude is half the distance you'd think it would be.

Period of Oscillation. Frequency of Oscillation.

$$\text{frequency} = \frac{1}{\text{period}} \tag{5.4}$$

$$f = 1/T \tag{5.5}$$

The unit of frequency is *hertz*, abbreviated Hz, and is defined by

$$1\,\text{Hz} = 1\,\frac{\text{repetition}}{\text{second}} \tag{5.6}$$

5.4.2 Properties of the Medium

Properties of the Medium: Density and Elasticity. How difficult is it to get the medium moving (mass from density). How easy is it for a disturbance to propagate to neighboring pieces (elasticity).

5.4.3 Properties of the Wave

The resulting wave, and its amplitude. Refer back to the section on energy on how the amplitude varies with propagation (either inverse-square law spreading, or attenuation).

Due to the periodicity of the source, the periodicity in time of motion in the medium. The consequent periodicity in space: the wavelength.

5.4.4 The Fundamental Equation

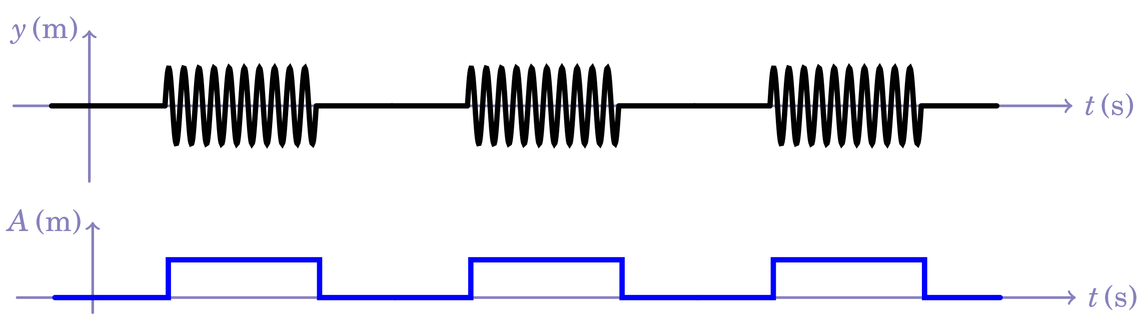
$$\lambda = v/f \tag{5.7}$$

$$\underbrace{\lambda}_{\text{Wavelength}} = \underbrace{v}_{\text{Wavespeed}} / \underbrace{f}_{\text{Frequency}} \tag{5.8}$$

5.4.5 Modulated Waves

The periodic wave, with constant amplitude and uniformly frequency of repetition, is the simplest category of waves to describe. But more useful – and common in the context of *Electrotherapy* – is the category of *modulated* waves.

A modulated wave is one with an amplitude that varies with time, a frequency that varies with time, or both. Most common is the wave with modulated amplitude. The simplest of that type, shown below, is a wave whose amplitude switches between being constant for an interval of time and being zero for the following interval, with this pattern of on/off repeating periodically.



In this modulated wave there are *two* repeating patterns: the rapid oscillations of the wave itself; and the longer period of the wave's amplitude being modulated.

5.4.6 Examples

5.5 Sound Waves

Strictly speaking, ***sound*** refers to the category of longitudinal waves in air that can be perceived by human hearing. However it is used to refer to the broader category of all longitudinal material waves, or "sound" in any material, like in soup, sand, or even inside the Sun. In the context of this course our focus is on sound in air, water, and the human body.

5.5.1 Sound in Air

The speed of sound in air varies with temperature and humidity. Aside: the speed of sound in a gas varies with the mass of the atoms or molecules that comprise the gas. The speed of sound is much faster in Helium, and much slower in Xenon.

Amplitude of Pressure fluctuations, relative to equilibrium atmospheric pressure. Magnitude of fluctuations: tiny!

Examples of frequency and wavelength in normal sounds.

5.5.2 Sound as a Longitudinal Wave

The displacement being parallel to the propagation. The difficulty to draw. Search the web for animations and movies.

Shockwaves

When the source moves faster than the speed of sound. When the amplitude exceeds the wavelength. Shockwaves.

5.5.3 Human Hearing

Anatomy. Mechanical motion to nerve impulses (transduction).

Sensitivity. Range of frequencies / wavelengths (floppiest to stiffest components). Range of pressure fluctuation amplitudes (threshold of detection to threshold of damage).

Cite Nobel lecture (Medicine, 1961) by Georg von Békésy on the mechanics of the inner ear.

Recommend the TED talk by Bobby McFerrin on singing musical notes that you don't know you know.

5.5.4 Sound in Water

The speed of sound in water varies with temperature and dissolved chemical content.

Hearing under water.

Shockwaves

When the source (or boundary) moves faster than the speed of sound. When the amplitude exceeds the wavelength. Shockwaves. Cavitation in ultrasound.

5.5.5 Sound in Other Materials

. [[tables of values of speed of sound in materials]]

Gasses: Air (different temperatures, humidity), Nobel gasses.

Liquids: Water (pure, different temperatures), water (Seawater, different temperatures), alcohol.

Solids: Ice, Plastic, wood, metal (Aluminum, Steel), concrete. (Difference in speeds dependent on deformation mode: shear, compression, torsion, etc.)

Human Biological context (ranges): Blood, fat, muscle, bone.

5.5.6 Ultrasound

Sounds we can't hear.

5.6 Energy in Sound Waves

[[illustration: a loudspeaker producing a sound]]

The kinetic energy in the oscillating air molecules and the work embodied in the change in pressure.

The idea that sound waves are generated by moving surfaces and received (detected) by surface that can move.

5.6.1 Definition: Sound Intensity

$$\text{Intensity} = \frac{\text{Power}}{\text{Area}} \tag{5.9}$$

$$\text{Energy} = \text{Intensity} \times \text{Area} \times \text{time} \tag{5.10}$$

$$\Delta E = I \times A \times \Delta t \tag{5.11}$$

At a position where the pressure fluctuation amplitude is Δp the corresponding intensity is

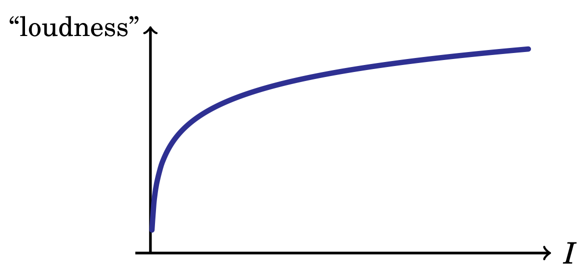
$$I = \frac{(\Delta p)^2}{2\rho v} \tag{5.12}$$

In the case of sound waves in air (where $\rho = 1.20\,\text{kg/m}^3$ is the density of air, and $v = 343\,\text{m/s}$ is the wavespeed in air) we get $\frac{1}{2\rho v} = 1.215 \times 10^{-3}\,\frac{\text{m}^2\text{s}}{\text{kg}}$. In the case of sound waves in water (where $\rho = 1\,000\,\text{kg/m}^3$ is the density of water, and $v = 1\,481\,\text{m/s}$ is the wavespeed in water) we get $\frac{1}{2\rho v} = 0.3376 \times 10^{-6}\,\frac{\text{m}^2\text{s}}{\text{kg}}$.

Difference in pressure required to achieve a target compression (do volume-change work) in gasses versus liquids.

5.6.2 The Decibel Scale

Sketch a graph of the subjective "loudness versus intensity" curve that motivates the use of the logarithm.



The quantitative measure β that models subjective loudness is called the ***sound level***, and is defined by:

$$\beta = (10\,\mathrm{dB})\log\left(I/I_0\right) \tag{5.13}$$

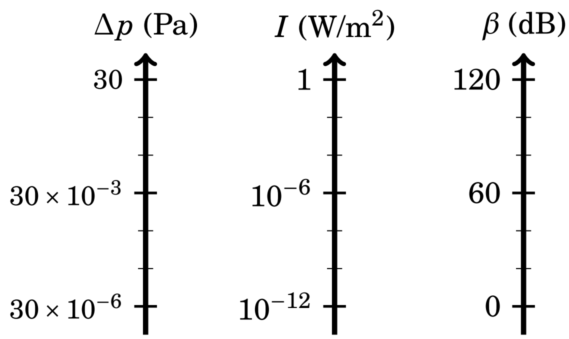
The logarithm returns a number, and the factor 10 dB expresses the level as a multiple of the unit decibel (dB) that measures level. The argument of the logarithm is the ratio of the intensity I of the sound wave to I_0 the *reference intensity* which is defined to be the exact quantity

$$I_0 = 1\times10^{-12}\,\mathrm{W/m^2} \tag{5.14}$$

which corresponds (approximately) to the quietest sound that an average person can hear.

DANGER: Some physics textbooks use the phrase “Sound Intensity” for the quantity of power over area, and use the phrase “Sound *Intensity Level*” for the measure of subjective loudness measured in decibels. The key is to look for the presence or absence of the word “Level” in the descriptor. In this text loudness in decibels is just “Sound Level”.

Comparing: amplitude of pressure fluctuation (pascals); sound intensity (watts per square metre); and sound level (decibels).



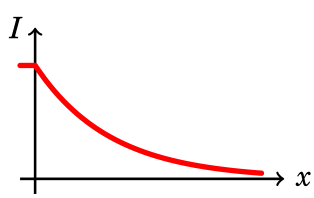
For the case of sound waves in air: the approximate correspondence between pressure fluctuation Δp, the corresponding sound intensity I, and sound level β.

[[diagram: “noise thermometer”]]

5.6.3 Attenuation

Experiment finds that, as a function of the distance x traveled into the medium, the intensity I of a wave decreases as

$$I(x) = I_1 \times 10^{-\alpha\, x/10\mathrm{dB}}$$



where “I_1” is the value of the intensity at the boundary $x = 0$ m where the wave enters the medium. The decrease in intensity is due to the material absorbing the energy of the wave, usually by the transformation of the wave’s mechanical oscillation into thermal energy in the medium. This dissipation of mechanical energy decreases the amplitude of the wave, and the wave is said to be ***attenuated***.

The quantity $\boldsymbol{\alpha}$ is called the *attenuation coefficient*. In the context of ultrasound applied to humans this coefficient is measured in units of decibels per centimetre (dB/cm), usually. The attenuation coefficient is a physical property of the medium: it depends upon the material, the material’s temperature, and the frequency of the wave(!). Generally the attenuation becomes much stronger with increasing frequency. (Low frequencies travel further than medium frequencies, while very high frequencies are absorbed over short distances.) The value of the attenuation coefficient is determined experimentally, though it can sometimes be determined theoretically. For us, it will be a value that is given, or a value to be solved-for from the relationship with intensity and distance.

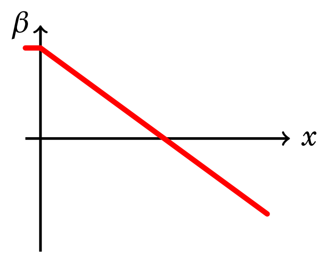
Since the initial sound intensity relates to its level through $I_1 = I_0 \times 10^{\beta_1/10\text{dB}}$ the attenuation, as written in the equation above, is equivalent to saying that the sound level is a *linear* function of distance the wave travels into the medium:

$$\beta(x) = \beta_1 - \alpha x \tag{5.15}$$

Because of this fact the attenuation coefficient is determined by measuring

$$\alpha = -\Delta\beta/\Delta x \tag{5.16}$$

The attenuation coefficient equals the slope of the plot of sound level as a function of depth into the material:



Imagine a wave of intensity I_1. Passing through a medium that absorbs energy (the wave is attenuated) this intensity will decrease to value I_2. How does the change in intensity relate to the change in level?

Consider this example: If 80% of the energy has been absorbed, then the intensity has dropped to 20% (which is one-fifth) of its initial value $I_2 = \frac{1}{5} I_1$. This can be written as $I_2/I_1 = 1/5$. The change in sound intensity level is thus

$$\Delta\beta = \beta_2 - \beta_1 \tag{5.17}$$

$$= (10\text{dB})\log\left(\frac{I_2}{I_0}\right) - (10\text{dB})\log\left(\frac{I_1}{I_0}\right) = (10\text{dB})\log\left(\frac{I_2}{I_0}\Big/\frac{I_1}{I_0}\right) = (10\text{dB})\log\left(\frac{I_2}{I_1}\right) \tag{5.18}$$

$$= (10\text{dB})\log(1/5) = (10\text{dB})(-0.699) = -6.99\,\text{dB} \tag{5.19}$$

This uses the fact that $\log A - \log B = \log(A/B)$.

In general, a drop in sound intensity to a fraction $I_2/I_1 = s < 1$ of its initial value leads to a change in sound level by an increment

$$\boxed{\Delta\beta = (10\text{dB})\log s} \tag{5.20}$$

Since $s < 1$ and thus $\log s < 0$, we will find that $\Delta\beta < 0$ dB. The negative sign in the defined $\alpha = -\Delta\beta/\Delta x$ is required because α is a positive quantity and $\Delta\beta < 0$ dB by attenuation.

5.7 Light & Electromagnetic Radiation

5.7.1 Light

The speed of light. Wavelength and frequency. Intensity and Energy.

Electromagnetic waves. Photons?

5.7.2 Invisible Light

Longer wavelengths. Infra-red. Microwaves. Radio waves.

Shorter wavelengths. Ultra-violet. X-Rays. Gamma rays.

5.7.3 "Radiation"

Particles versus Electromagnetic waves. Effects on biological tissue.

5.7.4 The Effects of Light on People

5.8 Examples

Chapter 6

Electricity

The word "electricity" maybe makes you think of tuning on a light in a room, or charging your cellphone. Powering a computer, moving the trains of the metro, or heating your house. Those are examples of what electricity can *do*. But what *is* electricity? This question is easy to pose, but very difficult to answer. This chapter will guide us towards being able to think physically about what electricity is, and what electricity can do.

There is just one thing that I will need you to never forget while studying this chapter:

DANGER: I am not a physiotherapist! :DANGER

In clinical settings you will be using electricity to apply physiothereputic treatments to people. You might think that it is unsafe to mix people with electricity. This is true in the same sense that there are dangers associated with driving a car. But when used properly, these technologies can be highly beneficial. You will learn proper and safe clinical techniques in your *Electrotherapy* course. In this course you will learn about what electricity *is*. Following the analogy of the car I will not be teaching you how to drive, I will be teaching you how the engine operates. Just never confuse anything in this course with advice on how to drive.

6.1 Electric Charges & Forces

What is "electricity"? The short answer is "the energy of interacting charges". To understand this we have to start with defining what charges are and how they interact.

Once we have done that we will able to talk about the *controlled* interaction of charges which form the basis of *electric circuits*. With the goal being preparation for your *Electrotherapy* course, in this course we need to get to a place where we can think about when *the patient* is part of the circuit.

6.1.1 Atoms: Electrons & the Nucleus

6.1.2 Electric Charge

In terms of electric force and sorting into categories of attract/repel. Sign convention.

For historical reasons the symbol used to denote charge is q. Usually lower-case q is used to denote the charge of atomic particles like electrons, protons, and ionized atoms, or the charge of microscopic particles like macromolecules or dust. (A little q for a little charge.) The upper-case Q is usually used to denote the charge of macroscopic objects. We will use ΔQ to denote the change

Difference between charge of the object's constituents and the net charge of the object.

Size of a charge relative to the macroscopic unit. The coulomb is huge in comparison to the charge of a single proton. (One coulomb is the charge of $6.241 \times 10^{+18}$ protons!) But we can't be too mad about that: The coulomb was decided upon A LONG TIME before the existence of atoms was agreed upon.

The charge of a single electron is

$$-1.602 \times 10^{-19}\,\mathrm{C} \tag{6.1}$$

The charge of a single proton is the positive of this.

6.1.3 Electric Forces

Referring back to section 1.2, where we mentioned how we rarely "see" electrical forces. Strength of electric force. Comparison with gravity.

6.1.4 Magnetic Forces

Interactions between moving charges. Magnetic forces.

6.2 Electric Current

6.2.1 Conductors, Insulators, and In-Between

Terminology: the conducting medium.

Current Carriers: electrons in orbitals, or ions in solution.

[[picture: mixture of ions of different sign flowing in opposite direction, net flow of "positive" current.]]

The "Conventional Current" is related to the sum of $q\vec{v}$ for all current carriers.

Electrons moving one way, and "current" flowing the other. Sad.

6.2.2 Measuring Current

An ampere is one coulomb per second:

$$1\,\text{ampere} = 1\,\frac{\text{coulomb}}{\text{second}} \tag{6.2}$$

$$1\,\mathrm{A} = 1\,\mathrm{C/s} \tag{6.3}$$

The abbreviation "amps" is often used in place of the word "amperes", most frequently when prefixes are present. For example, milliamperes are referred to as "milliamps".

The symbol used to denote current is I. This is unfortunate, since it will be confused with intensity that we used in context of sound waves, but is unavoidable. We are stuck with this historical convention. It will be up to you to be vigilant and know the meaning of the symbol by its context. This job will be much easier if you keep track of the units. (It should be impossible to mix-up a coulomb per second with a watt per square metre!)

Relation to a mole of electrons moving through a certain wire at a certain speed.

Measure of total charge transferred (ampere-seconds or ampere-hours).

Cases when the current flows in a single direction are referred to as Direct Current, abbreviated as DC. The current may change in amount, but if its direction does not change it is still considered DC. The cases where the direction does change are referred to as Alternating Current, abbreviated as AC.

6.2.3 Alternating Current (AC)

This looks like a wave, but it is not. This is a graph of our measurement of current a one place in the conductor. This is analogous to watching the motion of a single particle in the medium.

6.2.4 "Current Waveforms"

The current can vary with time in many more different, general ways.

6.2.5 Biological Currents

Dissolved ions. Charged species. Net flow of positive charge.

Ion channels. Concentration gradients and voltages.

Nerve impulses. Muscle activation.

6.2.6 Safety

Disruption of muscle activation. Disruption of nerve impulses. Damage of tissues by thermal energy. Disruption of heart rhythm. Disruption of brain function.

6.3 Electric Energy

Compare and contrast with the other forms of energy.

6.3.1 Sources of Electrical Power

Batteries convert chemical energy into electrical energy. Chemical reactions as reconfigurations of electrons in orbitals. Chemically induced separation of charge. (Importance in relation to biochemical reactions that are electro-chemical.)

Generators use magnetic forces to create currents. Mechanical work is transformed into electrical energy.

6.3.2 Electric "Potential"

The analogy between electrical potential energy and gravitational potential energy. The concept of *electrical potential* (which is distinct from electrical potential *energy*).

[[diagrams: The Analogy. Columns: force; field; energy; potential. Rows: Gravitational; Electrical.]]

When the electrical potential is different at different positions in or on an object we speak of the electrical potential difference *across* the object.

Differences in electrical potential are measured in units of *volts*. A volt is a joule per coulomb:

$$1\,\text{volt} = 1\,\frac{\text{joule}}{\text{coulomb}} \tag{6.4}$$

The symbol for potential is V. Sad.

6.3.3 Transfer versus Transform

Electrical current transfers electrical energy.

Current traveling from place to place transfers electrical energy. Current carriers interacting with the conducting medium transforms electrical energy into other forms of energy.

6.3.4 Electric Resistance

Dissipation of energy. Transformation of electrical energy into thermal energy.

The current carriers transport at least two forms of energy. They have kinetic energy because they have mass and are moving. They have electrical energy because they have electric charge and are moving across a difference in electrical potential.

The electric energy changes, the kinetic energy does not. The current is constant even as the electrical energy is being transformed. Current flowing through a resistor dissipates electrical energy into thermal energy. Current itself (the moving current carriers) are *not "used up"*. The current entering the resistor is the same as the current leaving the resistor.

The unit of resistance is the ohm, symbolized by the capital Greek letter omega: Ω.

There is a relationship between the current I through a conductor and the potential difference ΔV across it. The relation is known as *Ohm's Law*:

$$\Delta V = IR \tag{6.5}$$

The AC case. Definition of average that is non-zero. The RMS and Peak values, related. Emphasize that Ohm's Law is true for instantaneous values, so that it holds for the peak values and for the average.

6.3.5 Electric Power

The rate of energy transfer and/or transformation. Even though electrical energy is distinct from the other forms, it is still energy. The rate of electrical energy transfer or transformation is also measured in watts (joules per second).

$$P = I\,\Delta V = I^2 R \tag{6.6}$$

The AC case.

[[diagram: graph of AC power, and the way in which we get the "1/2" factor by chopping the tops off the "cos-squared" graph.]]

$$P_{\text{avg}} = \tfrac{1}{2} I_{\text{avg}}^2 R \tag{6.7}$$

6.3.6 Safety

Your skin, when dry, has a high electrical resistance (VALUE). But your insides are filled with highly conductive fluids.

6.4 Electric Circuits

When current travels along a wire, where does it go? It does not just arrive at the end of a wire and “pile up” or “fall out”!

The loop.

6.4.1 Circuit Elements

The source versus the elements of the circuit.

The battery and the resistor [electrical to thermal]. The battery and the light-bulb (or LED) [electrical to luminous]. The battery and the motor.

6.4.2 Series/Parallel

Multiple elements.

Terminology: Junction

Terminology: Branch

Series: The current is the same through each circuit element. The voltage across the group is the sum of the voltages across each element.

Parallel: The voltage is the same across each branch. The sum of currents entering a junction equals the sum of currents leaving the junction.

6.4.3 Power in a Circuit

Power put in by the source. Power divided across the elements. Power division across elements in series. Power division across elements in parallel.

6.4.4 Circuits with Time-Varying Current

"Electrical Waves" (cf section X on current above).

6.4.5 Safety

The role of "ground". "Short circuits". You should never be a part of the circuit!

6.5 Examples

